Incentive compatibility constraints and dynamic programming in continuous time

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Received 24 November 1998; received in revised form 19 July 1999; accepted 6 July 2000

Abstract

This paper is devoted to the study of infinite horizon continuous time optimal control problems with incentive compatibility constraints that arise in many economic problems, for instance in defining the second best Pareto optimum for the joint exploitation of a common resource, as in Benhabib and Radner [Benhabib, J., Radner, R., 1992. The joint exploitation of a productive asset: a game theoretic approach. Economic Theory, 2: 155–190]. An incentive compatibility constraint is a constraint on the continuation of the payoff function at every time. We prove that the dynamic programming principle holds, the value function is a viscosity solution of the associated Hamilton–Jacobi–Bellman (HJB) equation, and that it is the minimal supersolution satisfying certain boundary conditions. When the incentive compatibility constraint only depends on the present value of the state variable, we prove existence of optimal strategies, and we show that the problem is equivalent to a state constraints problem in an endogenous state region which depends on the data of the problem. Some economic examples are analyzed. © 2000 Elsevier Science S.A. All rights reserved.

Keywords: Optimal control; Incentive compatibility constraints; Value function; Dynamic programming; Second best Pareto optimum

JEL classification: C61; C73; E61; (AMS) 49-XX; 49L25; 90A16

1. Introduction

In this paper, we study optimal control problems in continuous time/infinite horizon with incentive compatibility constraints. We consider a classical infinite horizon optimal control
problem with a constraint on the continuation value of the plan at each time \( t \geq 0 \). The continuation value at each time \( t \) is constrained from below by a function of the state and of the control at time \( t \). The constraint can be interpreted from an economic point of view as an outside option/incentive compatibility constraint.

This type of constraint gives rise to two different problems: optimal stopping problems and optimal control problems with incentive compatibility constraints. In the first case, we have a \textit{positive} perspective: we want to study the optimal behavior of an economic agent in a dynamic setting allowing him/her at any time in the future to stop the process and to exercise an outside option which gives him/her a reward, which is a function of the state (termination payoff). In this setting, the set of admissible controls includes those violating the incentive compatibility constraint. The agent is allowed to get the termination payoff. In the second case, we have a \textit{normative} perspective. The point of view is one of a social planner who wants to characterize optimal contracts or second best solutions to dynamic problems under incentive compatibility constraints for the agents of the economy. The set of admissible controls does not include those violating the incentive compatibility constraint. The social planner looks for an optimal policy among the policies, which do not include the termination of the process. The goal is the definition of a social contract taking into account the fact that the agents can decide in the future to go out of the contract. Such an event is prevented by including the incentive compatibility constraint. This type of problems is analyzed in this paper.

Many economic problems can be put in this setting. They belong to two main classes of models of the so-called second best literature. The first type of problems comes from the study of differential games such as the exploitation of an exhaustible common resource. The second type of problems arises in analyzing policy making without full precommitment (time consistency problems). In Section 2, we will fully describe a second best problem arising in the analysis of a differential game for the exploitation of a common exhaustible resource.

It is well recognized that in differential games such as the exploitation of a common resource. \cite{Benhabib1994, Dockner1996, Tornell1992, Dutta1993a, Dutta1993b, Fudenberg1983}, capital accumulation \cite{Dockner1996}, pollution, voluntary provision of a public good \cite{Dockner1996}, the outcome of the noncooperative interaction obtained as a subgame perfect equilibrium or as trigger strategy equilibrium may be Pareto inefficient, i.e. the reward for the agent is smaller than the one obtained by a representative agent under perfect competition (the so-called \textit{tragedy of commons}). This result leads to the problem of designing contracts which are efficient among the subgame perfect equilibria \cite{Rustichini1992, Benhabib1996} and to the problem of designing contracts yielding at every time in the future a utility level higher than the one obtained according to a specific subgame perfect equilibrium. This type of problems can be formalized as the maximization of the utility of the representative agent under the constraint that at every time in the future, the continuation value of the consumption plan is greater than the utility obtained from the strategy of a subgame perfect equilibrium. Typically, the constraint is given by the value function of a control problem without constraints. The second class of models comes from optimal taxation problems in an intertemporal setting without full precommitment or in great generality from the analysis of an economy where there is a private sector and the government. The problem is the
definition of an optimal plan without full commitment at time zero (for instance, a tax plan made up of taxes by the government and saving decisions by the private sector) in such a way that the private sector and the government do not have an incentive to deviate at each time in the future (decisions are taken sequentially without commitment), (see Chari and Kehoe, 1990; Marcet and Marimon, 1992, 1996; Benhabib and Rustichini, 1997; Ireland, 1997) for some interesting examples.

The incentive compatibility constraint is endogenous to the model, it is usually given by a function only of the state and not of the control (in many cases, it is a value function of the associated unconstrained problem). The outside option can also be interpreted in some cases as a policy variable (fixed costs, royalties, taxes) or as an exogenous opportunity (a different investment opportunity). This type of constraints has been recently analyzed in discrete time in Marcet and Marimon (1996); Rustichini (1998a,b). In the first two papers, the problem has been studied by means of Lagrange multipliers, in the third one, through dynamic programming. Our paper provides a dynamic programming solution to the problem in the continuous time case. Some similarities between our characterization of the value function of the constrained problem and the one provided for the discrete time case in Rustichini, (1998a) can be noted (see Remark 4.4).

The optimal control problem is a state constraints problem with infinite horizon, a discounted objective function and an additional constraint on the continuation value for the plan. Such a constraint concerns the future of the trajectory and this “backward–forward” structure gives rise to nonstandard technical problems. Even simple one-dimensional problems (like the ones in Section 6) require a considerable amount of technical work. In fact, this type of problems has not been addressed in the present optimal control literature in continuous time: the classical tools of optimal control theory (like dynamic programming and maximum principle) are not available here.

In this paper, we first analyze a constraint described by a function of both the control and of the state and then we restrict our attention to the case of a function only depending on the state. We prove that the dynamic programming principle holds. This allows us to write the Hamilton–Jacobi–Bellman equation (HJB in the rest of the paper) associated with our problem and to prove, under some additional assumptions, that the value function of the constrained problem is a solution in the viscosity sense of the HJB equation. We cannot obtain uniqueness of solutions of the HJB equation in general, but we are able to characterize the value function as the minimal viscosity supersolution (satisfying suitable boundary conditions) of the equation. Restricting our attention to the case where the incentive constraint only depends on the state variable, we prove a result about existence (and uniqueness) of optimal strategies and then we prove that the above problem is equivalent to a state constraints problem in a region \( \mathcal{E} \) which is implicitly determined by the data of the problem. Then, the issues of determining the region \( \mathcal{E} \) and its topological properties are discussed. All of this allows us to adapt known results and techniques on state constraints problems so that we can study, in some cases, the properties of the optimal trajectories.

In Section 6, two simple examples are fully developed in order to make clear the main points of our second best analysis. The examples are characterized by a linear state equation, concave objective function, and an incentive compatibility constraint defined by a constant. The first example is the optimal saving problem, the second is the firm’s capital accumulation problem with adjustment costs. In the first example, if the interest rate is larger than the
discount rate, then both the value function and the optimal policy of the constrained problem coincide with those of the unconstrained problem, provided that the initial stock of capital is large enough. If the opposite condition holds, then the constrained value function is smaller than the unconstrained value function and the second best optimal control induces a smaller rate of consumption than the first best policy. The minimal stock of capital allowing existence of the second best policy is larger than in the first case. In the second example, as the constant describing the constraint goes up, we observe four different parameter regions. For a small constant (first region), the constraint is not binding and therefore, the unconstrained solution coincides with the constrained solution, for a higher constant (second region), the unconstrained solution is equal to the constrained solution but the initial stock of capital should be large enough to have a solution. As the constant is furthermore increased (third region), the investment rate is higher than the rate obtained in the first best case and an initial stock of capital larger than in the previous case is needed to have a solution. Finally, when the constant is beyond a certain level (fourth region), the problem becomes ill posed for every initial stock of capital.

Summing up in the above two examples, we have that an incentive compatibility constraint has two effects: it restricts the state region for which a solution exists and it induces a higher rate of investment. The optimal policy foresees a stationary level of the state variable when the incentive constraint becomes binding.

The paper is organized as follows. In Section 2, we present a motivating example and then we introduce the mathematical framework of the problem. The basic theory addressing the main technical problems is developed in Sections 3–5. Sections 3 and 4 are concerned with the dynamic programming principle and the HJB equation, respectively, in a general framework. To improve the readability of these sections, we relegated the proofs of some technical results to the Appendix. Section 5 deals with the case when the incentive constraint depends only on the state (which occurs in main economic examples). Here, we prove the equivalence theorem (Section 5.1) that allows to reduce the problem to a state constraints problem in an endogenous region $E$, and then (in Section 5.2), we discuss the properties of $E$. Moreover in Sections 5.3 and 5.4, we prove the upper-semicontinuity of the value function and the existence of optimal strategies. In Section 6, we analyze two examples.

2. The problem

In this section, we introduce the class of models considered in our analysis. Section 2.1 presents the second best Pareto optimum problem for the exploitation of a common exhaustible resource as introduced in Benhabib and Radner (1992). Section 2.2 presents a general mathematical formulation of the model; an equivalent formulation, which will be useful in studying some of its properties, is given in Section 2.3. Section 2.4 contains technical assumptions that will be used in the paper.

2.1. Exploitation of a common exhaustible resource

The example comes from Benhabib and Radner (1992), where a two-agent ($i = 1, 2$) differential game for the exploitation of an exhaustible resource is analyzed. Player’s $i$
The rate of consumption \( c_i(t) \) is constrained from above by a constant \( \bar{c} \). The stock of the resource evolves as follows:

\[
\dot{y}(t) = \begin{cases} 
  n(y(t)) - c_1(t) - c_2(t), & \text{when } y(t) > 0, \\
  0, & \text{when } y(t) = 0,
\end{cases}
\]

(1)

and the initial condition is \( y(0) = y_0 \). The function \( n \) is strictly concave, differentiable, \( n(0) = n(y^0) = 0 \) where \( y^0 > 0 \), \( n(y) > 0 \) for \( y \in (0, y^0) \) and \( n(y) < 0 \) for \( y > y^0 \). Let \( \bar{c} \) be large and assume that there exists a \( \bar{y} > 0 \) such that \( n'(\bar{y}) = \rho \), where \( \rho \) is the agents’ discount factor. Agent \( i \) maximizes the discounted sum of consumption

\[
\rho \int_0^\infty e^{-\rho t} c_i(t) \, dt.
\]

(2)

The maximization of the sum for the discounted consumption of the two agents \( c(t) = c_1(t) + c_2(t) \) gives us the first best solution (Pareto optimum) of the differential game (efficient equilibrium). Let \( V(y) \) be the value function associated with this solution.

This solution is a good (social) contract for the agent of the economy under full pre-commitment, i.e. the game is played one shot at \( t = 0 \) and then the agents do not change their strategies. If this is not the case, then agents may be tempted to change their strategies as time goes on. Allowing the agents to change their strategies, we end up with Markov Nash equilibria or subgame perfect equilibria. In the paper, the authors were interested in “exploring the possibility of sustaining efficient equilibria by trigger strategies, that is by the players’ credible threats to revert to a Markov Nash equilibrium whenever any player deviates from the efficient path”, see Benhabib and Radner (1992), p. 165.

In Benhabib and Radner (1992) and Rustichini (1992), it is shown that the worst Markov Nash equilibrium is the one where the two agents consume at the maximum rate \( \bar{c} \) (extreme strategy). The point is the following: assuming that the agents agree to follow the first best solution, is this solution immune to the threat of switching to the extreme strategy? The answer is positive if the value function associated with the first best solution is higher than the reward obtained by adopting the extreme strategy for every value of the state along the trajectory of the first best solution.

In Benhabib and Radner (1992), it is assumed that agents detect a departure from a target strategy with a delay \( \tau > 0 \). For small \( \tau \), they assume that the consumption in the period \( \tau \) before defection is detected is \( \delta = \bar{c} \tau \). It is shown that for a fixed \( \delta \), the value of defection as \( \tau \to 0 \) (i.e. forcing the maximum rate of consumption \( \bar{c} \to +\infty \) is

\[
D(y) = \begin{cases} 
  \frac{\rho(y + \delta)}{2}, & y \geq \delta, \\
  \frac{\rho y}{\rho y}, & 0 < y < \delta.
\end{cases}
\]

(3)

The authors compare the value function \( V(y) \) with \( D(y) \). They show that for \( \delta \) and \( y \) sufficiently high \( D(y) > V(y) \), therefore for a large enough stock of the resource defection will be attractive, for \( y \) small defection will be attractive even if \( D(y) < V(y) \) provided that for some \( y' > y \) along the trajectory of the first best solution, we have \( D(y') > V(y') \). Therefore, it is not enough to compare \( V(y) \) with \( D(y) \) for some \( t \), the inequality should be checked for every state point along the trajectory. For a particular technology, the authors exactly determine the state region of sustainability of the first best solution.
The second best problem for this game, as introduced by Rustichini (1992), is then the one of determining the supremum in

$$\rho \int_0^\infty e^{-\rho t} c(t) \, dt$$

for a representative agent subject to:

$$\dot{y}(t) = \begin{cases} n(y(t)) - c(t), & \text{when } y(t) > 0, \\ 0, & \text{when } y(t) = 0, \end{cases} \quad y(0) = y_0,$$

(5)

to the obvious state and control constraints $y(t) \geq 0$, $c(t) \geq 0$ for every $t \geq 0$, and to the additional incentive constraint

$$\rho \int_0^\infty e^{-\rho s} c(s) \, ds \geq e^{-\rho t} D(y(t)).$$

The solution to this problem is a contract, which is immune from defection by the agents at every time in the future. Of course, the solution of the second best problem will be the solution of the first best problem if the last one is sustained by the threat.

This paper aims at analyzing this kind of problems (see also Rustichini, 1992 where a similar problem, but with strictly concave utility is considered). The example above will not be fully analyzed below, some of its results are described at the end of Section 6.

2.2. Mathematical formulation of the problem

Let $C \subset \mathbb{R}^d$ and let $C$ be the set of all functions $c: \mathbb{R}^+ \rightarrow C$ that are measurable and locally integrable. Given $c \in C$, consider the state equation

$$\begin{cases} \dot{x}(s) = f(x(s), c(s)); & s \geq 0 \\ x(0) = x_0, & x_0 \in \mathbb{R}^n, \end{cases}$$

(6)

where $f$ is a function that is Lipschitz continuous in $x$, uniformly with respect to $c$. In the standard framework, the positive real half-line describing the domain of the function $c$ represents the time dimension of the problem. The function $c$ is the control, $c(s) \in \mathbb{R}^d$ is the value of the control function at time $s$. The dimension of the state variable $x$ is $n$ while $d$ is the dimension of the control variable. Let $x(t; x_0, c) \in \mathbb{R}^n$ denote the solution of Eq. (6) at time $t \geq 0$ given the control $c \in C$ and the initial condition $x_0 \in \mathbb{R}^n$. The solution always exists given the assumptions specified above.

Given a set $A \subset \mathbb{R}^n$, define for every $x_0 \in A$ the set $\tilde{C}_A(x_0)$ as the set of controls $c \in C$ such that $x(t; x_0, c) \in A$ for every $t \geq 0$. $A$ represents the state constraint, for example, a positivity state constraint requires $A$ to be the positive orthant in $\mathbb{R}^n$.

We consider two different control problems: an unconstrained and a constrained problem.
Let us first take a continuous objective function \( f_0: A \times C \mapsto \mathbb{R} \). Given \( x_0 \in \mathbb{R}^n \), the unconstrained problem is just a classical state constraints optimal control problem with infinite horizon: maximize the functional

\[
J(x_0; c) = \int_0^{+\infty} e^{-\rho t} f_0(x(t; x_0, c), c(t)) \, dt
\]

over all controls \( c \in C_A(x_0) \) such that \( J(x_0, c) \) is well defined (see, e.g. Hartl et al., 1995 for the maximum principle approach and Soner, 1986; Capuzzo-Dolcetta and Lions, 1990; Cannarsa et al., 1991; Ishii and Koike, 1996; Soravia, 1997b) and Bardi and Capuzzo-Dolcetta, 1998, Chap. IV) for the dynamic programming approach to this kind of problems.

The unconstrained value function is defined as

\[
V_u(x_0) \overset{\text{def}}{=} \sup_{c \in C_A(x_0)} J(x_0; c),
\]

\( V_u(x_0) = -\infty \) when \( C_A(x_0) = \emptyset \). Under suitable controllability assumptions (e.g. the set \( A \) has a \( C^{1,1} \) boundary and \( \forall x_0, \exists c_0 \) such that \( f(x_0, c_0), n(x_0) < -\varepsilon \) where \( n(x_0) \) is the outward normal vector at \( x_0 \) and \( \varepsilon > 0 \) is independent of \( x_0 \), (see Cannarsa et al., 1991, Remark 4.7), we have that \( C_A(x_0) \neq \emptyset \forall x_0 \in A \). We make the following assumption.

**Assumption 2.1.**

(i) The sets \( A \) and \( C \) are closed and convex.

(ii) \( f \) is continuous and there exists a constant \( M > 0 \) such that

\[
\begin{align*}
|f(x_1, c) - f(x_2, c)| & \leq M|x_1 - x_2| & \forall x_1, x_2 \in A, \forall c \in C \\
|f(x, c)| & \leq M(1 + |x| + |c|) & \forall x \in A, \forall c \in C.
\end{align*}
\]

(iii) \( f_0 \) is continuous and uniformly continuous in \( x \), uniformly in \( c \).

To ensure that the value function \( V_u \), is always finite, we also assume the following.

**Assumption 2.2.**

For every \( x_0 \in A \) and every admissible control strategy \( c \in C_A(x_0) \), we have \( |J(x_0; c)| \leq M \left( |x_0| \right) \) where \( M \) is a suitable nondecreasing function on \( \mathbb{R}^+ \).

Further assumptions will be needed to deal with the incentive constrained problem described below. Some of them, like Assumption 2.3(ii) and (iii) will also imply “good” properties for the unconstrained problem.

To begin with the constrained problem, we define \( C(x_0) \) as the set of controls \( c \in C_A(x_0) \) such that the following constraint is satisfied for almost every \( t \geq 0 \):

\[
\int_t^{+\infty} e^{-\rho s} f_0(x(s; x_0, c), c(s)) \, ds \geq e^{-\rho t} D(x(t; x_0, c), c(t)),
\]

where \( D: A \times C \mapsto \mathbb{R} \) is a suitable continuous function. The constraint Eq. (9) can be interpreted as an incentive compatibility and/or rationality constraint. The control plan should guarantee at each time \( t \) a residual payoff higher than a function of the state and of the control at time \( t \).
The constrained problem consists of maximizing the functional Eq. (7) over all controls $c \in \mathcal{C}(x_0)$. The constrained value function is defined as

$$V(x_0) \equiv \sup_{c \in \mathcal{C}(x_0)} J(x_0; c),$$

(10)

$V(x_0) = -\infty$ when $\mathcal{C}(x_0) = \emptyset$. Obviously, $\mathcal{C}(x_0) \subset \mathcal{C}_A(x_0)$ so that we have

$$V(x_0) \leq V_u(x_0) \quad \forall x_0 \in A.$$

An optimal strategy (usually denoted by $c^*$) for the above problem is an admissible strategy for which the supremum in Eq. (10) is attained. The corresponding state trajectory will be denoted by $x^*$ and the pair $(x^*, c^*)$ will be called an optimal couple.

Let us define the set $E = \{ x \in A : V_u(x) \geq D_0(x) \} \subset A$.

2.3. Equivalent formulation

Both the unconstrained and the constrained problems can be rewritten by setting

$$w(t; x_0, c) = \int_t^{+\infty} e^{-\rho(s-t)} f_0(x(t; x_0, c), c(s)) ds.$$  

(11)

The function $w(\cdot; x_0, c)$ is then the unique solution of the equation

$$\begin{cases}
\dot{w}(t) = \rho w(t) - f_0(x(t; x_0, c), c(t)) & t \geq 0 \\
\lim_{t \to +\infty} e^{-\rho t} w(t) = 0.
\end{cases}$$

The unconstrained problem becomes: maximize $J(x_0, c) = w(0)$ under the constraints

$$\begin{cases}
\dot{x}(t) = f(x(t), c(t)) & t \geq 0 \\
\dot{w}(t) = \rho w(t) - f_0(x(t), c(t)) & t \geq 0 \\
x(0) = x_0 \\
\lim_{t \to +\infty} e^{-\rho t} w(t) = 0 \\
x(t) \in A.
\end{cases}$$

(12)

while the constrained problem is obtained by adding the requirement that

$$w(t) \geq D(x(t; x_0, c), c(t)) \quad t \geq 0.$$  

We observe that the above problem is not a standard state constraints problem since the new state variable $w$ solves a backward equation with terminal condition at $t = +\infty$. This indicates the main difficulty we have to face when dealing with an incentive constraint: not only is it a nonlocal constraint (which would be eliminated by adding the new state variable) but also a constraint on the future value of the plan: it is a backward–forward problem. The above formulation is equivalent to the first one and will be useful in some proofs (see Section 5). We will see in Section 5 that our incentive constrained problem is also equivalent to the
state constraints problem in the region \( E \), i.e. to the problem of maximizing the functional Eq. (7) under the state Eq. (6) and the state constraints \( E \). For this problem, for every \( x \in E \), the set of admissible strategies is \( C_E(x) \supseteq C(x) \) (with possible strict inclusion).

2.4. Technical assumptions

We now collect and explain various technical assumptions that will be used in the paper. They are dictated by the motivating examples of Section 6. We are forced to work with unbounded control and space regions as well as unbounded state and cost functions \( f \) and \( f_0 \). Because of this, the assumptions become complicated, technical, and non-uniform. However, they are necessary to obtain results on existence (and uniqueness) of optimal strategies, characterization of the value function, and topological properties of the region \( E \). In the bounded cases, they can be simplified or avoided entirely since it is then easy to invoke compactness (see, e.g. Soravia, 1997b). The assumptions below are not needed to prove the dynamic programming principle in Section 3 and the equivalence theorem in Section 5.

We denote by \( B(x_0, r) \) the closed ball in \( \mathbb{R}^n \) centered at \( x_0 \) with radius \( r \) and we say that a function \( \omega : \mathbb{R}^+ \to \mathbb{R}^+ \) is a modulus if it is continuous, subadditive, nondecreasing, and such that \( \lim_{a \to 0} \omega(a) = 0 \). Moreover, given \( \varepsilon > 0 \) and \( x \in E \), we say that a control strategy \( c \in C(x) \) (respectively, in \( C_A(x) \)) is \( \varepsilon \)-suboptimal for the constrained (respectively, unconstrained) problem if \( J(x; c) > V(x) - \varepsilon \) (respectively, \( J(x; c) > V_u(x) - \varepsilon \)).

**Assumption 2.3.**

(i) For every \( x \in E \), the set of pairs \((f_0(x, C), f(x, C))\) is closed and convex.

(ii) For every \( x_0 \in E \), \( T > 0 \) there exists \( r > 0 \) and a modulus \( \omega \) such that, for every \( y \in B(x_0, r) \cap E \), there exists a control strategy \( c_y \in C_E(y) \) such that

\[
\int_{t_1}^{t_2} \left[ |f_0(x(t); y, c_y)| + |f(x(t); y, c_y)| \right] dt \leq \omega(|t_1 - t_2|),
\]

\( \forall 0 \leq t_1, t_2 \leq T \),

\( \int_{t_1}^{+\infty} e^{-\rho t} |f_0(x(t); y, c_y)| dt \to 0 \) uniformly for \( y \in B(x_0, r) \cap E \). (13)

(iii) For every \( x_0 \in E \), \( T > 0 \) and for every \( \varepsilon > 0 \) there exists \( r > 0 \) and a modulus \( \omega \) such that, for every \( y \in B(x_0, r) \cap E \), there exists an \( \varepsilon \)-suboptimal control strategy \( c_{y, \varepsilon} \in C_E(y) \) such that

\[
\int_{t_1}^{t_2} \left[ |f_0(x(t); y, c_{y, \varepsilon})| + |f(x(t); y, c_{y, \varepsilon})| \right] dt \leq \omega(|t_1 - t_2|),
\]

\( \forall 0 \leq t_1, t_2 \leq T \),

\( \int_{t_1}^{+\infty} e^{-\rho t} |f_0(x(t); y, c_{y, \varepsilon})| dt \to 0 \) uniformly for \( y \in B(x_0, r) \cap E \). (15)
For every \( x_0 \in E, \ T > 0 \) and for every \( \varepsilon > 0 \), there exists \( r > 0 \) and a modulus \( \omega \) such that, for every \( y \in B(x_0, r) \cap E \), there exists an \( \varepsilon \)-suboptimal control strategy \( c_{y, \varepsilon} \in C(y) \) such that
\[
|x(t_1, y, c_{y, \varepsilon}) - x(t_2; y, c_{y, \varepsilon})| \leq \omega(|t_1 - t_2|) \quad \forall t_1, t_2 \in [0, T].
\] (17)

**Remark 2.4.**

1. Assumption 2.3(i) is the so-called Roxin condition, which is a standard condition used in many existence theorems (like the Fillipov Theorem, see Cesari, 1983).
2. Assumption 2.3(ii), (iii), (iv) is expressed in quite a complicated form. They guarantee a little compactness needed in proofs. More precisely, Assumption 2.3(ii) gives some uniform integrability with respect to the set of admissible strategies and is helpful in proofs of properties of \( E \); Assumption 2.3(iii) produces some uniform integrability with respect to \( \varepsilon \)-optimal strategies and is needed to prove existence of optimal strategies; Assumption 2.3(iv) gives uniform continuity of \( \varepsilon \)-optimal trajectories and is used in proofs of results characterizing the value function in Section 5.
3. Since in many cases the set \( E \) is unknown, it is not easy in general to check the above assumptions. However, if Assumption 2.3(i) is true for \( x \in A \) (or for \( V_u(x) \geq D_0(x) \)), it is true for \( x \in E \); moreover, other Assumption 2.3(ii), (iii), (iv) can be checked indirectly or using ad hoc arguments in the examples of Section 6.

### 3. The dynamic programming principle

In this section, we prove the dynamic programming principle for the constrained problem, i.e. we will establish the following theorem:

**Theorem 3.1.** Let Assumptions 2.1(ii),(iii) and 2.2 be satisfied. Then, for every \( T > 0 \), we have
\[
V(x_0) = \sup_{c \in C(x_0)} \inf_{T} J_T(x_0; c),
\]
where
\[
J_T(x_0; c) def = \int_0^T e^{-\rho t} f_0(x(t; x_0, c), c(t))dt + e^{-\rho T} V(x(T; x_0, c)).
\]

The dynamic programming principle is an important result since it establishes a link between the control problem and the HJB equation that is considered in the next section. It will allow us to characterize the value function as a solution of the HJB equation and thus, use the machinery of partial differential equations to study the control problem. Theorem 3.1 is a standard result for the unconstrained problem. Its proof is based on a principle of shifting and pasting of controls and can be found for instance in Fleming and Soner (1993). This is no longer available in the incentive constrained case because the juxtaposition of two controls admissible for two different points may produce a control that is no longer admissible since it may fail to satisfy the incentive constraint. We overcome this problem
in Lemma 3.3. We begin with two necessary technical lemmas whose proofs are deferred to the Appendix.

**Lemma 3.2.** Let Assumptions 2.1(ii),(iii) and 2.2 be satisfied. Given any \( T > 0, x_0 \in E, c \in C(x_0) \), the control \( c_T \) defined as \( c_T(s) = c(T + s) \) with \( s > 0 \) belongs to \( C(x(T; x_0, c)) \).

The above lemma states that given an initial state \( x_0 \) and a control satisfying the constraint Eq. (9), this control restricted to the interval \([T, +\infty)\) satisfies the constraint Eq. (9) starting from \( x(T; x_0, c) \) for every \( T > 0 \).

**Lemma 3.3.** Let Assumptions 2.1(ii),(iii) and 2.2 be satisfied. Given \( x_0 \in E, T > 0 \) and \( c \in C(x_0) \), take a control trajectory \( c_T \in C(x(T; x_0, c)) \) and define the new control

\[
    c_1(t) = \begin{cases} c(t) & t \in [0, T) \\ 
    \tilde{c}_T(t - T) & t \in [T, +\infty) \end{cases}
\]

(18)

If \( J(x(T; x_0, c); \tilde{c}_T) \geq J(x(T; x_0, c); c) \), then \( c_1 \in C(x_0) \).

**Proof of Theorem 3.1.** First we prove that

\[
    V(x_0) \geq \sup_{c \in C(x_0)} J_T(x_0; c).
\]

Let \( c \in C(x_0) \). It follows from Lemma 3.2 that \( c_T \in C(x(T; x_0, c)) \). If this control is optimal, then it follows from the definition of the value function that \( J_T(x_0; c) = J(x_0; c) \leq V(x_0) \). If \( c_T \) is not optimal starting at \( x(T; x_0, c) \), then for an arbitrary \( \varepsilon > 0 \), we can find a control \( c_{\varepsilon, T} \in C(x(T; x_0, c)) \) (the so-called \( \varepsilon \)-suboptimal control) such that

\[
    V(x(T; x_0, c)) < \varepsilon + J(x(T; x_0, c); c_{\varepsilon, T}),
\]

and

\[
    J(x(T; x_0, c); c_{\varepsilon, T}) \geq J(x(T; x_0, c); c_T).
\]

(19)

Then, by Lemma 3.3, the control \( c_{\varepsilon} \) defined as

\[
    c_{\varepsilon}(t) = \begin{cases} c(t) & t \in [0, T) \\ 
    \tilde{c}_{\varepsilon}(t - T) & t \in [T, +\infty) \end{cases}
\]

belongs to \( C(x_0) \) and we have

\[
    J_T(x_0; c) \leq J(x_0; c_{\varepsilon}) + e^{-\rho T} \varepsilon \leq V(x_0) + \varepsilon,
\]

where the first inequality comes from Eq. (19) and the second from the definition of the value function. This can be proved for every \( \varepsilon > 0 \) and therefore, \( J_T(x_0; c) \leq V(x_0) \) for \( x_0 \in C(x_0) \).

To complete the proof, we have to prove the opposite inequality. We observe that for every \( \varepsilon > 0 \), there exists \( c_{\varepsilon}' \in C(x_0) \) such that

\[
    V(x_0) < J(x_0; c_{\varepsilon}') + \varepsilon
\]

and, by Lemma 3.2,
\[
J(x_0; c'_x) = \int_0^T e^{-\rho t} f_0(x(t; x_0, c'_x), c'_x(t))dt + e^{-\rho T} J(x(T; x_0, c'_x); c'_{xT})
\]
\[
\leq \int_0^T e^{-\rho t} f_0(x(t; x_0, c'_x), c'_x(t))dt + e^{-\rho T} V(x(T; x_0, c'_x))
\]
\[
= J_T(x_0; c'_x).
\]
Therefore
\[
V(x_0) < J_T(x_0, c'_x) + \epsilon,
\]
which implies that
\[
V(x_0) \leq \sup_{c \in C(x_0)} J_T(x_0; c).
\]

4. The Hamilton–Jacobi equation

As we already mentioned in the previous section, the dynamic programming principle can be recast in terms of a partial differential equation that is satisfied by the value function. This is the so-called HJB equation that can be thought of as the infinitesimal version of the principle of optimality. The derivation of the HJB equation (i.e. the fact that the value function is its solution) in the unconstrained case is now standard and rigorous even when the value function is not smooth (see Fleming and Soner, 1993). It uses the notion of a generalized solution viscosity solution. We refer the reader to Fleming and Soner (1993) and Crandall et al. (1992) for an overview of the theory of viscosity solutions. It allows merely continuous (or even discontinuous) functions to be solutions of partial differential equations and is based on the principle of “differentiation by parts” that replaces possibly non-existing derivatives of a solution \(U\) by derivatives of smooth functions \(\varphi\) at every point \(x\) such that \(U-\varphi\) has a local maximum or minimum at \(x\).

For problems with state constraints, the associated HJB equations have to be understood in a special sense. They have been first investigated by Soner (Soner, 1986) and later by many authors (Capuzzo-Dolcetta and Lions, 1990; Ishii and Koike, 1996; Soravia, 1997b). In this paper, we adapt the approach of Ishii and Koike (1996). Their definition of solution is based on the earlier definition of Soner and uses a so-called inward Hamiltonian \(H_{in}\) that is defined with the help of special admissible controls. The definition of \(H_{in}\) reflects the fact that admissible controls depend on the initial point \(x\), something that shows in the derivation of the HJB equation. Some of the arguments employed below are similar to those used in Ishii and Koike (1996) and Soner (1986).

Define \(A(x)\) to be the set of all \(c \in C\) such that there exists \(r > 0\) such that for every \(y \in E \cap B(x, r)\) there exists \(c(\cdot) \in \bar{C}(y)\) such that \(c(t) = c\) for \(t \in [0, r]\). We will assume that
\[
A(x) \neq \emptyset
\]
for every \(x \in E\). In simple words, we can say that the set \(A(x)\) is closely related to the set of \(c \in C\) that are starting points of admissible strategies for the constrained problem and we
can think of it as the set of instantaneous control strategies that are effectively doable when we are at state $x$. The sets $E$ in Examples in sections 6.1 and 6.2 satisfy this assumption.

Let

$$H(x, p) = \sup_{c \in C} \{(f(x, c), p) + f_0(x, c)\},$$

and

$$H_{in}(x, p) = \sup_{c \in \mathcal{A}(x)} \{(f(x, c), p) + f_0(x, c)\}.$$  

Since $C$ may be unbounded, both Hamiltonians may take infinite values. However, being a supremum of continuous functions, $H$ is lower-semicontinuous in $p$. Moreover, $H$ is uniformly continuous in $x$, uniformly for bounded $p$ such that $H(x, p)$ is finite.

We now give the Ishii–Koike definition of viscosity solution applied to our case. (We point out that the Ishii–Koike definition is a little stronger than the one introduced in a similar context in Soravia (1997b). Let $U^*$ and $U_*$ denote, respectively, the upper- and the lower-semicontinuous envelope of $U : E \to \mathbb{R}$.

**Definition 4.1.** A locally bounded function $U : E \to \mathbb{R}$ is a viscosity subsolution (respectively, supersolution) of the equation

$$\rho U - H(x, DU) = 0$$

in $E$ if whenever $U^* - \varphi$ has a local maximum (respectively, $U_* - \varphi$ has a local minimum) at $x$ relative to $E$, where $\varphi \in C^1(E)$, then

$$\rho U^*(x) - H(x, D\varphi(x)) \leq 0$$

(respectively,

$$\rho U_*(x) - H_{in}(x, D\varphi(x)) \geq 0.$$  

A function $U$ is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

In the above definition $\varphi \in C^1(E)$ means that there exists an open set $\Omega$ such that $E \subset \Omega$ and $\varphi \in C^1(\Omega)$.

**Theorem 4.2.** Let $\mathcal{A}(x) \neq \emptyset$ for every $x \in E$, and let 2.1(ii),(iii), 2.2, and 2.3(iv) be satisfied. Let $H$ be upper-semicontinuous at every point $(x, p)$ such that $H(x, p) \neq +\infty$. Then, the value function $V$ defined in Eq. (10) is a viscosity solution of Eq. (21) in $E$.

**Proof.** Let $V^* - \varphi$ have a local maximum at $x_0 \in E$. We may assume that the maximum is 0. If $H(x_0, D\varphi(x_0)) = +\infty$, we are done. If not, we will argue by contradiction. If Eq. (22) is not satisfied at $x_0$, then the continuity of $H$ at $(x_0, D\varphi(x_0))$ implies that there exist $r, \delta > 0$ such that if $x \in E$, $|x - x_0| < r$ then

$$\rho \varphi(x) - H(x, D\varphi(x)) \geq \delta.$$  

(24)
It follows that there exist \( x_\epsilon \in E, x_\epsilon \to x_0 \) as \( \epsilon \to 0 \), and almost optimal controls \( c_\epsilon \in C(x_\epsilon) \) as in Assumption 2.3(iv) such that
\[
\epsilon^2 \geq -\int_0^\epsilon e^{-\rho t} f_0(x(t; x_\epsilon, c_\epsilon), c_\epsilon(t)) \, dt - \epsilon e^{-\rho \epsilon} \varphi(x(\epsilon, x_\epsilon, c_\epsilon)) + \varphi(x_\epsilon)
\]
\[
= -\int_0^\epsilon e^{-\rho t} (f_0(x(t; x_\epsilon, c_\epsilon), c_\epsilon(t)) - \rho \varphi(x(t, x_\epsilon, c_\epsilon)))
+ (f(x(t; x_\epsilon, c_\epsilon), c_\epsilon(t)), D\varphi(x(t; x_\epsilon, c_\epsilon)))) \, dt.
\]
Therefore, for some \( t < \epsilon \), we have
\[
\rho \varphi(x(t; x_\epsilon, c_\epsilon)) - f_0(x(t; x_\epsilon, c_\epsilon), c_\epsilon(t)) - (f(x(t; x_\epsilon, c_\epsilon), c_\epsilon(t)), D\varphi(x(t; x_\epsilon, c_\epsilon))) \leq 2\epsilon.
\]
However, this contradicts Eq. (24) since by Assumption 2.3(iv) \( |x(t; x_\epsilon, c_\epsilon) - x_0| < \epsilon \) for small \( \epsilon \).

Let now \( V^* - \varphi \) have a local minimum (equal to 0) at \( x_0 \in E \). Let \( c \in A(x_0) \). Let \( r \) be as in the definition of \( A(x_0) \). Let \( c_\epsilon(t) \in C(x) \), for \( x \in B(x_0, r) \), be such that \( c_\epsilon(t) = c \) for \( t \in [0, \epsilon] \), and \( x(t; x, c) \in E \) for \( t \in [0, \epsilon] \). For \( 0 < \epsilon \leq r \), let \( x_\epsilon \in E \cap B(x_0, r) \) be such that
\[
V(x_\epsilon) < \varphi(x_\epsilon) + \epsilon^2,
\]
and \( \lim_{\epsilon \to 0} x_\epsilon = x_0 \). The dynamic programming principle gives
\[
V(x_\epsilon) \geq \int_0^\epsilon e^{-\rho t} f_0(x(t; x_\epsilon, c_\epsilon), c_\epsilon(t)) \, dt + e^{-\rho \epsilon} V(x(\epsilon, x_\epsilon, c_\epsilon)).
\]
Therefore
\[
\epsilon^2 \geq \int_0^\epsilon e^{-\rho t} f_0(x(t; x_\epsilon, c_\epsilon), c_\epsilon(t)) \, dt + e^{-\rho \epsilon} \varphi(x(\epsilon, x_\epsilon, c_\epsilon)) - \varphi(x_\epsilon)
\]
\[
= \int_0^\epsilon e^{-\rho t} (f_0(x(t; x_\epsilon, c_\epsilon), c_\epsilon(t)) - \rho \varphi(x(t, x_\epsilon, c_\epsilon)))
+ (f(x(t; x_\epsilon, c_\epsilon), c_\epsilon(t)), D\varphi(x(t; x_\epsilon, c_\epsilon)))) \, dt.
\]
It follows that for some \( y_\epsilon = x(t_\epsilon; x_\epsilon, c_\epsilon) \), we have
\[
2\epsilon \geq f_0(y_\epsilon, c) - \rho \varphi(y_\epsilon) + (f(y_\epsilon, c), D\varphi(y_\epsilon)).
\]
Moreover, since \( c_\epsilon(t) = c \) for \( t \in [0, r] \), we have \( \lim_{\epsilon \to 0} y_\epsilon = x_0 \). Thus, passing to the limit as \( \epsilon \to 0 \), we obtain
\[
\rho V^*(x_0) - (f(x_0, c), D\varphi(x_0)) - f_0(x_0, c) \geq 0.
\]
Since \( c \) is an arbitrary element of \( A(x_0) \), the claim follows. \[\square\]

Unfortunately viscosity solutions of Eq. (21) are not unique in general. A simple example of non-uniqueness is exhibited in Remark 6.2. The following characterization of the value
function seems to be the best possible. The optimality of this characterization is discussed in Remark 4.4. The Proof of Proposition 4.3 is moved to the Appendix.

**Proposition 4.3.** Let \( u \) be a locally bounded and globally bounded from below, lower-semicontinuous viscosity supersolution of Eq. (21) in \( E \). Let Assumptions 2.1(ii),(iii) and 2.2 be satisfied and let \( H = H_{in} \) in \( \text{int}E \). If \( u \geq V \) on \( E \setminus \text{int}E \) then \( u \geq V \) in \( E \).

**Remark 4.4.** When \( E \) is open, then \( E \setminus \text{int}E = \emptyset \) and the value function is then the minimal supersolution. This characterization is similar to the one obtained in Soravia (1997b), Theorem 4.5. However, the assumptions used in Soravia (1997b), Theorem 4.5 are different from ours and moreover, we deal with unbounded solutions. For the discrete time case, a characterization of the value function as the maximal subsolution of the discrete time HJB equation is obtained in Rustichini (1998a). Such a characterization does not seem to be possible without further restrictions in the continuous time case and in fact, the characterization provided by Proposition 4.3 seems to be the best possible in the generality stated there. In Remark 6.2, we present explicit examples of two functions \( V_1 \) and \( V_2 \) such that \( V_1 < V_2 \). \( V_1 \) is the value function, however, they both are viscosity solutions of the associated HJB equation. To obtain a characterization on the other side (as the maximal subsolution in a certain family) in this setting one should at least consider subsolutions with a prescribed growth at infinity.

**Remark 4.5.** To apply Proposition 4.3, one needs to know when \( H = H_{in} \) in \( \text{int}E \). A relatively easy criterion can be obtained in the case that \( D \) is independent of \( c \). It is then enough to have \( V_e > D \) in \( \text{int}E \), i.e. that for every \( x \in \text{int}E \) there exist \( \delta > 0, \epsilon > 0 \) such that

\[
V(y) \geq D(y) + \delta \quad \text{for } y \in B(x, \epsilon). \tag{25}
\]

To see this, we take any \( c \in C \). We need to show that \( c \in \mathcal{A}(x) \). Let \( c_1(t) = c \) and let \( r \) be small enough such that if \( y \in B(x, r) \) then \( y(t; y, c_1) \in B(x, \epsilon/2) \) for \( t \in [0, r] \),

\[
\int_0^r e^{-\mu t} |f_0(y(t; y, c_1), c_1(t))| dt < \frac{\delta}{3},
\]

and

\[
|D(y(t; y, c_1)) - D(y)| = \frac{\delta}{6} \quad \text{for } t \in [0, r].
\]

Let \( c_2 \in C(y(r; y, c_1)) \) be such that

\[
\int_r^{+\infty} e^{-\mu t} f_0(y(t; y(r; y, c_1), c_2), c_2(t)) dt > V(y(r; y, c_1)) - \frac{\delta}{3}
\geq D(y(r; y, c_1)) + \frac{2\delta}{3}.
\]

Define

\[
c(t) = \begin{cases} c_1(t) & t \in [0, r] \\ c_2(t - r) & t > r. \end{cases}
\]
It is now clear that $c() \in C(y)$ since if $T \in [0, r]$ then
\[
\int_T^{+\infty} e^{-\rho t} f_0(y(t; y, c), c(t)) dt > D(y(r; y, c_1)) + \frac{2\delta}{3} - \frac{\delta}{3} > D(y(T; y, c)).
\]
Thus $c \in A(x)$.

A sufficient condition for Eq. (25) is for instance that $D$ is a smooth function satisfying
\[
\rho D(x) - H_{in}(x, DD(x)) < 0 \text{ in } int E.
\]
We notice that if $V$ is lower-semicontinuous at every point of $E \setminus int E$ and $H=H_{in}$ in $int E$, then $V_\delta$ is a lower-semicontinuous supersolution of Eq. (21), $V = V_\delta$ on $E \setminus int E$, and so if the assumptions of Proposition 4.3 are satisfied, then $V \leq V_\delta$ in $E$, i.e. $V$ is lower-semicontinuous. Therefore, also, if the assumptions of Proposition 5.7 are satisfied, the value function $V$ is continuous in $E$.

5. State dependent incentive compatibility constraints

In this section, we will assume that the constraint map $D$ only depends on the state variable $x$. As we have observed in the Introduction and in Section 2.1, many incentive compatibility constraints for economic problems are of this type. We will address the following points:

- in Section 5.1, we will prove that the problem can be viewed as a state constraints problem in a region $E$ to be determined from the data of the problem;
- in Section 5.2, we will discuss properties of $E$ and outline methods to find it in one-dimensional examples;
- in Section 5.3, we will present some regularity properties of the value function; and
- in Section 5.4, we will prove a result about existence of optimal strategies.

5.1. Equivalence with a state constraints problem

Consider two new control problems in the region $E$. Given $x_0 \in E$ and $c \in C$, a number $\tau$ such that $x(\tau; x_0, c) \in \partial E$ and $x(t; x_0, c) \in E$ for every $t \leq \tau$ will be called an exit time of the trajectory from $E$ (there can be many $\tau$ like this: we will denote by $\Theta(x_0, c)$ the set of all exit times). The first problem consists in minimizing the payoff function
\[
J_{ET}(x_0; c, \tau) = \int_0^\tau e^{-\rho t} f_0(x(t), c(t)) dt + e^{-\rho \tau} D(x(\tau)).
\]
Denote its value function by
\[
V_{ET}(x_0) = \sup_{c \in C, \tau \in \Theta(x_0, c)} J_{ET}(x_0; c, \tau).
\]
The second problem is the state constraints problem of maximizing the payoff functional Eq. (7) in the narrower region $E$. For $x_0 \in E$, define the value function
\[
V_{SC}(x_0) = \sup_{c \in C_E(x_0)} J(x_0; c).
\]
The following result is stated assuming that Assumption 2.1 holds, but it remains true also if the sets $A$ and $C$ are open.

**Theorem 5.1.** Let Assumption 2.1 and 2.2 hold. Then, for every $x_0 \in E$, we have

$$V_{SC}(x_0) = V(x_0) \leq V_{ET}(x_0).$$

If $E$ is closed, then

$$V(x_0) = V_{ET}(x_0).$$

**Proof.** Let us consider the first statement. Given an $x_0 \in E$, it is easy to prove that $V_{SC}(x_0) \geq V(x_0)$, in fact, we have that $C(x_0) \subset C_E(x_0)$. This follows from the fact that $c \in C(x_0)$ implies that $x(t; x_0, c) \in E$ for every $t \geq 0$. Indeed, by contradiction, if $x(t; x_0, c) \notin E$ then $C(x(t; x_0, c)) = \emptyset$, which is impossible by Lemma 3.2.

To prove that $V_{SC}(x_0) \leq V(x_0)$, we show that for every control strategy $c \in C_E(x_0)$, there exists another strategy $\tilde{c} \in C(x_0)$ such that $J(x_0; \tilde{c}) \geq J(x_0; c)$. If $c \in C(x_0)$, then there is nothing to prove. Assume that this is not the case. Then, there exists $\tilde{t} \geq 0$ such that

$$\int_{\tilde{t}}^{+\infty} e^{-\rho t} f_0(x(t; x_0, c), c(t))dt < e^{-\rho \tilde{t}} D(x(\tilde{t}; x_0, c)).$$

Let $t_0$ be the infimum of the set of such $\tilde{t}$. If $t_0 = 0$ then $J(x_0; c) \leq D(x_0) \leq J(x_0; \tilde{c})$ for every $\tilde{c} \in C(x_0)$, where $C(x_0) \neq \emptyset$ because $x_0 \in E$, and there is nothing more to prove. If $t_0 > 0$ then the continuity of $f_0$ and $D$ implies

$$\int_{t_0}^{+\infty} e^{-\rho s} f_0(x(s; x_0, c), c(s))ds = e^{-\rho t_0} D(x(t_0; x_0, c))$$

(26)

and, for $t \leq t_0$,

$$\int_{t}^{+\infty} e^{-\rho s} f_0(x(s; x_0, c), c(s))ds \geq e^{-\rho t} D(x(t_0; x_0, c)).$$

Consider a control strategy $c_{t_0} \in C(x(t_0; x_0, c))$ that satisfies $J(x(t_0; x_0, c), c_{t_0}) \geq J(x(t_0; x_0, c), c)$. Such a control exists since $x(t_0; x_0, c) \in E$ and Eq. (26) holds.

Then, by Lemma 3.3, the control strategy $\tilde{c}$ defined as

$$\tilde{c}(t) = \begin{cases} c(t) & t \in [0, t_0) \\ c_{t_0}(t - t_0) & t \in [t_0, +\infty) \end{cases}$$

belongs to $C(x_0)$. Moreover, by Eq. (26), the definition of $\tilde{c}$, and the fact that $c_{t_0} \in C(x(t_0; x_0, c))$, it follows that

$$J(x_0; \tilde{c}) - J(x_0; c) = \int_{t_0}^{+\infty} e^{-\rho s} f_0(x(s; x_0, \tilde{c}), \tilde{c}(s))ds - e^{-\rho t_0} D(x(t_0; x_0, \tilde{c})) > 0$$

proving the claim that $V_{SC}(x_0) \leq V(x_0)$. 
\( V_{ET}(x_0) \) is greater than or equal to the values of the other two value functions. This comes from the observation that both \( C(x_0) \) and \( C_E(x_0) \) are subsets of the set of controls that we consider in the exit time problem.

To prove the second statement, assume that \( E \) is closed. Above, we have shown that \( V_{ET}(x_0) \geq V_{SC}(x_0) = V(x_0) \). It remains to be proven that given a control strategy \( c \) with exit time \( \tau < +\infty \) there exists \( \tilde{c} \in C_E(x_0) \) such that

\[
J_{ET}(x_0; c) = \int_0^\tau e^{-\rho s} f_0(x(s; x_0, c), c(s))ds + e^{-\rho \tau} D(x(\tau; x_0, c)) \leq J(x_0; \tilde{c}).
\]

To find such \( \tilde{c} \), we first consider a control strategy \( c^* \in C(x(\tau; x_0, c)) \) (such \( c^* \) always exists since, by the closedness of \( E \), we have \( x(\tau; x_0, c) \in E \)). We then define

\[
\tilde{c}(t) = \begin{cases} 
  c(t) & t \in [0, \tau) \\
  c^*(\tau - s) & t \in [\tau, +\infty).
\end{cases}
\]

Since \( C(x(\tau; x_0, c)) \subset C_E(x(\tau; x_0, c)) \), it is clear that \( \tilde{c} \in C_E(x_0) \). Moreover, since \( c^* \in C(x(\tau; x_0, c)) \),

\[
J(x(\tau; x_0, c); c^*) = \int_0^{+\infty} e^{-\rho s} f_0(x(s; x(\tau; x_0, c), c^*), c^*(s))ds \geq D(x(\tau; x_0, c))
\]

so that, by the definition of \( c^* \),

\[
J(x_0; \tilde{c}) = \int_0^{+\infty} e^{-\rho s} f_0(x(s; x_0, \tilde{c}), \tilde{c}(s))ds \\
= \int_0^\tau e^{-\rho s} f_0(x(s; x_0, c), c(s))ds \\
+ \int_\tau^{+\infty} e^{-\rho s} e^{-\rho \tau} f_0(x(s - \tau; x(\tau; x_0, c), c^*), c^*(s - \tau))ds \\
= \int_0^\tau e^{-\rho s} f_0(x(s; x_0, c), c(s))ds + e^{-\rho \tau} J(x(\tau; x_0, c); c^*) \\
\geq J_{ET}(x_0; c, \tau)
\]

which completes the proof.

**Remark 5.2.** In light of the above theorem, when \( D \) is independent of \( c \), the HJB equation related to the incentive compatibility constrained problem coincides with the one obtained for the problem constrained in the region \( E \). However, this does not help resolve the issue of uniqueness of solutions. We also observe that the above theorem gives a condition under which the state constraints problem and the exit time (or stopping time) problem have the same value function.

**Remark 5.3.** From the Proof of Theorem 5.1 (more precisely from its second paragraph), it follows that, given \( x \in E \), if \( c^* \in C_E(x) \) is optimal for the state constraints problem in \( E \), then \( c^* \in C(x) \) and so it is also optimal for the incentive constrained problem. This fact will be useful in the analysis of examples in Section 6.
5.2. The set $E$

We now investigate the set $E$. We obtain some of its geometrical and topological properties that will later be used to determine the set in economic models studied in Section 6. We start by giving a sufficient condition for the closedness of $E$. Obviously, as recalled in Section 2, $E \subseteq \{ x \in X : V_n(x) \geq D(x) \} \subseteq A$, and $V_n(x) \geq V(x)$ for $x \in E$.

**Proposition 5.4.** If Assumptions 2.1, 2.2, and 2.3(i),(ii) are satisfied, then $E$ is closed.

**Proof.** We use some arguments related to the well-known “Filippov existence theorem” (see, e.g. Cesari, 1983, Chap. 9).

First, we consider a sequence $(x_n)_{n \in \mathbb{N}} \subseteq E$ converging to a point $x_\infty \in A$. Let $c_n \in C_E(x_n)$ be an admissible control for $x_n$ provided by Assumption 2.3(ii). Then we take the equivalent form of our problem by introducing the variable $w$ as in Eq. (12) and for every $n \in \mathbb{N}$ we consider the trajectory $(x_n, w_n) := (x, w)(\cdot; x_n, c_n) \in W^{1,1}_{1}\{(0, +\infty); \mathbb{R}^{n+1}\}$.

Assumptions 2.2 and Eq. (13) give equiboundedness and equicontinuity of the sequence $\{(x_n, w_n)\}_{n \in \mathbb{N}}$ and equintegrability of the sequence $\{(x_n)(w_n)(\cdot)\}_{n \in \mathbb{N}}$. We now use the Ascoli–Arzelà Theorem to say that, along a suitable subsequence, we have the uniform convergence on bounded subsets of $[0, +\infty)$:

$$x_n \to x_\infty(\cdot), \quad w_n \to w_\infty(\cdot),$$

where $x_\infty \in C^0([0, +\infty); \mathbb{R}^n)$ and $w_\infty \in C^0([0, +\infty)\mathbb{R})$. Moreover, by the Dunford–Pettis Theorem (see, e.g. Cesari, 1983, p. 329) we obtain that, along a further subsequence if necessary, we have the $L^1$-weak convergence on bounded subsets of $[0, +\infty)$:

$$(x_n)' \to \xi(\cdot), \quad (w_n)' \to \eta(\cdot),$$

where $\xi_\infty \in L^1_{1}[0, +\infty); \mathbb{R}^n]$ and $\eta \in L^1_{1}([0, +\infty); \mathbb{R})$. By integrating, we get

$$x_n(t) = x_\infty + \int_0^t (x_n)'(s)ds$$

so that, passing to the limit we obtain

$$x_\infty(t) = x_\infty + \int_0^t \xi(s)ds$$

so that $x_\infty \in W^{1,1}_{1}([0, +\infty); \mathbb{R}^n)$.

On the other hand, setting $z_n(t) = e^{-\int_0^t f_0(x_n(t), c_n(t))}$, we have $(z_n)'(t) = e^{-\int_0^t f_0(x_n(t), c_n(t))}$ and so, by Eq. (14), $\int_0^t |(z_n)'(t)|dt \to 0$ as $t \to +\infty$, uniformly for $n \in \mathbb{N}$. We now claim that $(z_n)'$ converges weakly in $L^1([0, +\infty); \mathbb{R})$ to the function $t \to \zeta(t) = e^{-\int_0^t \eta(t)}$. Indeed, by the weak convergence of $w_n$ above, we have that

$$(z_n)' \to \zeta,$$
$L^1$-weakly on bounded subsets of $[0, +\infty)$. Then, for any $t_1 > 0$
\[
\int_{t_0}^{t_1} |\xi(t)|\,dt \leq \liminf_{n \to +\infty} \int_{t_0}^{t_1} |(z^n)'(t)|\,dt
\]
but the right-hand side is bounded uniformly for $t_1 > 0$ by Eq. (14), so that $\xi \in L^1([0, +\infty); \mathbb{R})$. Moreover, given $g \in L^\infty([0, +\infty); \mathbb{R})$, we have that for every $t_1 > 0$
\[
\left| \int_0^{+\infty} g(t)(z^n)'(t) - \xi(t) \right| \,dt \leq \int_0^{t_1} g(t)[|(z^n)'(t)| + |\xi(t)|] \,dt
\]
and the claim follows by taking first $t_1$ sufficiently big and then letting $n \to +\infty$. Having the weak convergence of $(z^n)'$ in $L^1([0, +\infty); \mathbb{R})$ allows us to pass to the limit as $n \to +\infty$ in the relation
\[
z^n(t) = \int_t^{+\infty} (z^n)'(s) \,ds
\]
obtaining (defining $z^\infty(t) = e^{-\rho t} w^\infty(t)$)
\[
z^\infty(t) = \int_t^{+\infty} \xi(s) \,ds,
\]
so that $z^\infty \in W^{1,1}(0, +\infty); \mathbb{R})$ and $(z^\infty)' = \xi$ which implies that $w^\infty \in W^{1,1}_{loc}(0, +\infty); \mathbb{R})$, and $(w^\infty)' = \eta$.

At this point, we can apply the so-called closure theorem (see, e.g., Cesari, 1983, p. 299 or p. 340) to say that the trajectory $(x^\infty, w^\infty)$ satisfies the differential inclusion
\[
(x^\infty)'(t) \in f(x^\infty(t), C); \quad (w^\infty)'(t) \in \rho w^\infty - f_0(x^\infty(t), C).
\]
The hypotheses needed to apply the closure theorem of Cesari (1983), p. 299 require that the set valued map $E \times \mathbb{R} \mapsto E \times \mathbb{R}$
\[
(x, w) \mapsto F(x, w) \overset{\text{def}}{=} (f(x, C), \rho w - f_0(x, C))
\]
satisfies the so-called property $(Q)$. This property holds when (see again Cesari, 1983, p. 293, 8.5.iv) Assumption 2.3(i) is satisfied and the map in Eq. (27) is upper semicontinuous by set inclusion. This property (Cesari, 1983, p. 291) says that, for every $(x_0, w_0) \in E \times \mathbb{R}$ and every $\epsilon > 0$, there exists $\delta > 0$ such that
\[
\bigcup_{(x, w) \in B(x_0, w_0, \delta)} F(x, w) \subset \{(x, w) \text{ such that } d((x, w), F(x_0, w_0)) < \epsilon\}
\]
and this is clearly automatically satisfied when Assumption 2.1 holds. We remark that the result (Cesari, 1983, p. 293, 8.5.iv) holds only for autonomous set valued maps. This is the reason why we did not use $z$ as a state variable.
Now, we can apply a theorem about existence of measurable selectors for solutions of differential inclusions (see Cesari, 1983, p. 278, 8.2.iii) which gives that there exists an admissible strategy $c^\infty$ such that

$$x^\infty(t) = x(t; x^\infty, c^\infty), \quad w^\infty(t) = \int_t^{+\infty} e^{-\rho(s-t)} f_0(x^\infty(s), c^\infty(s)) \, ds.$$  

It is now enough to prove that all the constraints are satisfied for $(x^\infty, w^\infty)$, but this is an immediate consequence of the uniform convergence stated above.

The following proposition about the convexity of $E$ is elementary and will be useful in treating some examples. It also holds when the sets $A$ and $C$ are not closed.

**Proposition 5.5.** Let Assumptions 2.1 and 2.2 hold. Assume moreover that $A$ is convex, $f_0$ is concave in $(x, c)$, $f$ is linear, and $D$ is convex. Then $E$ is convex.

**Proof.** Let $x_1, x_2 \in E$, and let $c_1 \in C(x_1), c_2 \in C(x_2)$. Take $\lambda \in (0, 1)$ and set $x_\lambda = \lambda x_1 + (1-\lambda)x_2, c_\lambda = \lambda c_1 + (1-\lambda)c_2$. First of all, we observe that $x_\lambda \in A$ since $A$ is convex. Then, it is enough to prove that $c_\lambda \in C(x_\lambda)$. By the linearity of $f$, we have that for every $t \geq 0$

$$x(s; x_\lambda, c_\lambda) = \lambda x(s; x_1, c_1) + (1-\lambda)x(s; x_2, c_2)$$

so that, by the concavity of $f_0$

$$\int_t^{+\infty} e^{-\rho(s-t)} f_0(x(s; x_\lambda, c_\lambda), c_\lambda(s)) \, ds$$

$$\geq \lambda \int_t^{+\infty} e^{-\rho(s-t)} f_0(x(s; x_1, c_1), c_1(s)) \, ds$$

$$+ (1-\lambda) \int_t^{+\infty} e^{-\rho(s-t)} f_0(x(s; x_2, c_2), c_2(s)) \, ds$$

and, by the admissibility of $c_1, c_2$ and the convexity of $D$,

$$\geq \lambda D(x(t; x_1, c_1)) + (1-\lambda)D(x(t; x_2, c_2)) \geq D(x(s; x_\lambda, c_\lambda)).$$

which gives the claim. \qed

For the one-dimensional case, we also have the following result whose proof we omit since its arguments are similar to those of the Proof of Proposition 5.5.

**Proposition 5.6.** Assume that $d = 1$ and that Assumptions 2.1 and 2.2 hold.

1. Assume moreover that $A$ is convex, $f_0$ is concave in $(x, c)$ and increasing in $x$, $f$ is concave, $D$ is convex and decreasing. Then, $E$ is convex.

2. Assume moreover that $f_0$ is increasing in $x$ and $D$ is decreasing. Then $x \in E$ implies that $y \in E$ for every $y \geq x, y \in A$.  

The results above help identify the region $E$. We will be able to do this in the one-dimensional cases studied in Sections 6.1 and 6.2. Here is a brief explanation of the main ideas that will be used to do this. We will show that one can apply Propositions 5.4, 5.5, and 5.6. If so, we will know that $E$ can only be a closed interval (possibly empty) or a half line contained in the set $\{ V_u \geq D \}$. It will only remain to find its endpoints. Then we will check if and when the optimal strategies for the unconstrained problem are admissible for the constrained one. This can give $E = \{ V_u \geq D \}$ immediately, and provide a useful information. If the above are not enough, we will use the maximum principle for state constraints problems on $E$ to determine the optimal strategy as a function of the extrema of $E$ and then verify the incentive constraints for such strategies. This method could be adopted in solving the problem presented in Section 2.1 when the utility function is concave.

5.3. Regularity of the value function

**Proposition 5.7.** Let Assumptions 2.1, 2.2, and 2.3(iii) be satisfied. Then $V$ is upper semi-continuous.

**Proof.** Let $x_n \to x_0$. We will prove that $\limsup_{n \to +\infty} V(x_n) \leq V(x_0)$, i.e. that $\limsup_{n \to +\infty} V_{SC}(x_n) \leq V_{SC}(x_0)$ thanks to Theorem 5.1. First, we observe that we can restrict our attention to the case when $x_n \in E$ for every $n \in \mathbb{N}$ and $x_0 \in E$. We then take a sequence of control strategies $c_n \in C_E(x_n)$ such that

$$J(x_n, c_n) > V(x_n) - \frac{1}{n}$$

and let $(x^n, w^n) = (x, w)(x_n, c_n)$ be the associated trajectories. By applying the same arguments as those used in the Proof of Proposition 5.4, we obtain that $(x^n, w^n)$ converges uniformly along a subsequence to an element $(x^\infty, w^\infty) \in W^{1,1}_{loc}([0, +\infty); \mathbb{R}^{n+1})$ which is still associated to an admissible strategy $c_\infty$. Then

$$\limsup_{n \to +\infty} V(x_n) \leq \limsup_{n \to +\infty} w^n(0) + \frac{1}{n} = w^\infty(0) \leq V(x_0),$$

which gives the claim. \qed

We cannot expect the continuity of the value function in general. In fact, also in the state constraints case (which is a special case of our problem when $D \leq J$ for all admissible strategies), there are examples where the value function is not lower semicontinuous, even under the assumptions of the above proposition (see, e.g. (Soravia, 1997b, Ex. 4.3). However, using Remark 4.5, we can prove continuity in some cases (see Section 6). Another special case is discussed below.

**Proposition 5.8.** Let the assumptions of Proposition 5.5 be satisfied. Then the value function is concave.

**Proof.** Let $x_1, x_2 \in E$ and $\lambda \in (0, 1)$ be given. By the definition of the value function, we have that, for every $\varepsilon > 0$ there exists $c_1 \varepsilon \in C(x_1)$, $c_2 \varepsilon \in C(x_2)$ such that

$$\lambda V(x_1) + (1 - \lambda) V(x_2) < \lambda J(x_1, c_1 \varepsilon) + (1 - \lambda) J(x_2, c_2 \varepsilon) + 2\varepsilon,$$
and by concavity of $f_0$

$$\begin{align*}
\lambda V(x_1) + (1 - \lambda) V(x_2) & \geq \int_0^{+\infty} e^{-\mu t} f_0(\lambda x(t); x_1, c_{1\varepsilon}) \\
&\quad + (1 - \lambda) f_0(x(t); x_2, c_{2\varepsilon}) + \lambda c_{1\varepsilon}(t) + (1 - \lambda) c_{2\varepsilon}(t) \, dt + 2\varepsilon.
\end{align*}$$

Then, by the linearity of $f$, setting $x_{\lambda} = \lambda x_1 + (1 - \lambda) x_2$, $c_{\lambda\varepsilon} = \lambda c_{1\varepsilon} + (1 - \lambda) c_{2\varepsilon}$ and recalling that $c_{\lambda\varepsilon} \in \bar{C}(x_{\lambda})$, we have thanks to Proposition 5.5

$$\begin{align*}
\lambda V(x_1) + (1 - \lambda) V(x_2) & \leq \int_0^{+\infty} e^{-\mu t} f_0(x(t); x_3, c_{3\varepsilon}) + c_{3\varepsilon}(t) \, dt + 2\varepsilon \leq V_\varepsilon(x_3) + 2\varepsilon.
\end{align*}$$

The claim follows by the arbitrariness of $\varepsilon$.

For the one-dimensional case, we have the following result whose proof is similar to the Proof of Proposition 5.8.

**Proposition 5.9.** Assume that $d = 1$ and that Assumptions 2.1 and 2.2 hold.

1. Assume moreover that $A$ is convex, $f_0$ is concave in $(x, c)$ and increasing in $x$, $f$ is concave, $D$ is convex and decreasing. Then $V$ is concave.

2. Assume moreover that $f_0$ is increasing in $x$ and $D$ is decreasing. Then $x, y \in E$ and $y \geq x$ implies that $V(y) \geq V(x)$.

5.4. An existence result for optimal strategies

We state and sketch the proof of an existence result for optimal strategies.

**Theorem 5.10.** Let Assumptions 2.1, 2.2, and 2.3(iii) be satisfied. Then for every $x_0 \in E$, there exists an optimal strategy for the constrained problem.

**Proof.** We again use a modification of the well-known Fillipov Theorem adapted to our problem in the formulation given at the end of Section 2. We only sketch the main points of the proof since they are based on the arguments used in the Proof of Proposition 5.4.

Let $x_0 \in E$, $(c_n)_{n \in \mathbb{N}} \subset \bar{C}(x_0)$ be a maximizing control sequence for our problem starting at $x_0$ and satisfying Assumption 2.3(iii). Let then $(x^n, w^n) = (x, c(\cdot); x_0, c_n)$ be the associated trajectories. Arguing as in the Proof of Proposition 5.4, we obtain that $(x^n, w^n)$ converges uniformly along a subsequence to $(x^\infty, w^\infty) \in W^{1,1}_{\text{loc}}([0, +\infty); \mathbb{R}^{n+1})$ for which there is an admissible strategy $c_{\infty}$. Then

$$V(x_0) = \lim_{n \to +\infty} J(x_0; c_n) = \lim_{n \to +\infty} w^n(0) = w^\infty(0) \leq V(x_0),$$

and so $c_{\infty}$ is optimal for the state constraints problem in $E$. The claim follows by applying Remark 5.3.

It seems possible in some cases to characterize optimal strategies using the HJB equation showing in particular that they can be written as $c^\varepsilon(t) = \gamma(x^\varepsilon(t))$ for a suitable map $\gamma$, etc.
or taking $\gamma$ to be multivalued and replacing the equality by an inclusion. Such a strategy $c^*$ is called an optimal feedback strategy or, according to Benhabib and Radner (1992), a stationary strategy. However since the value function does not even have to be differentiable, such a characterization would have to involve rather sophisticated tools of nonsmooth analysis. Therefore, we will not attempt to discuss this issue here. An interested reader can consult Bardi and Capuzzo-Dolcetta (1998) for some results in this direction.

Remark 5.11. If $f_0$ is independent of $x$, strictly concave in $c$, and the unconstrained problem is closed under convex combinations, then the optimal strategy is unique (see Section 6).

6. Examples

In this section, we analyze two simple economic examples. The first one is a consumption problem, the second one is an investment problem with adjustment costs. In both cases, we assume that the incentive compatibility constraint is given by a constant. The point of view is the one of a social planner who wants to analyze the optimal capital accumulation in an economy where the private sector can stop the process at any time in the future and to go abroad if the future utility/profit is smaller than a given constant. The social planner wants to define the optimal capital accumulation path among the policies, which do not lead the agents to break the contract terminating the accumulation process (second best solution). The incentive compatibility constraint is set equal to a constant to fully carry out the analysis. In a policy perspective, we can think of the constant as a control variable (tax) managed by the government at $t = 0$, e.g. a sunk cost to run the firm, etc.

We will not study the control problem of finding the second best solution for the motivating example introduced in Section 2.1. Assumptions 2.1 and 2.2 are satisfied there however due to the unboundedness of the control set $\mathcal{C}$ and the nonlinear nature of the state equation, it is not obvious if Assumption 2.3 hold. Moreover, since the assumptions of Propositions 5.5 and 5.6 are clearly not satisfied, not much can be said at this point about the set $E$. Nevertheless, some theorems apply, most notably the dynamic programming principle and the equivalence theorem (Theorem 5.1) so the problem is equivalent to a state constraint problem in the region $E$. Something more can be said in the case of nonlinear utility function studied in Rustichini (1992). However, since a full analysis of Example in section 2.1 (also in the case of Rustichini, 1992) requires additional results, we will not perform it here postponing it to a future work.

Instead, we will present a complete analysis of the next two examples, where it is possible to obtain satisfactory results applying the whole machinery developed in this paper.

6.1. Consumption with a fixed positivity constraint on the value function

Let us consider the following problem:

$$\max \int_{0}^{+\infty} e^{-\rho t} \frac{c(t)^{\alpha}}{\alpha} dt, \quad 0 < \alpha < 1,$$

$$\dot{x}(t) = ax(t) - c(t), \quad x(0) = x_0,$$
subject to the usual constraints $c \geq 0, x \geq 0$ and to the following incentive compatibility constraint on the payoff function:

$$
\int_0^{+\infty} e^{-\rho(s-t)} \frac{C^0(s)}{\alpha} ds \geq D(x(t; x_0, c)) \quad \forall t \geq 0.
$$

(28)

The function $D$ is assumed to be equal to a positive constant $D_0 > 0$.

This is a classical optimal consumption problem, $c$ denotes consumption and $x$ denotes the stock of capital, $a$ is the coefficient of the linear technology, $a > 0$, i.e. the instantaneous interest rate. The constraint Eq. (28) says that the continuation value should be greater than a fixed positive level $D_0$. At every time, the agent wants a utility in the future bigger than this constant.

The value function of the unconstrained problem is finite for every starting point $x_0$ if and only if $\alpha < 1$ (see, e.g. Fleming and Soner, 1993, p. 29). We will assume that this holds from now on. The HJB equation is

$$
\rho V(x) - H(x, DV(x)) = 0, \quad x \geq 0,
$$

where the maximum value Hamiltonian $H$ is given by

$$
H(x, p) = \sup_{c \geq 0} \left\{ (ax - c)p + \frac{c^\alpha}{\alpha} \right\} = axp + H_0(p)
$$

with

$$
H_0(p) = \begin{cases} 
+\infty & \text{if } p \leq 0 \\
1 - \frac{\alpha}{a} & \text{if } p > 0.
\end{cases}
$$

The maximum is reached at $c^* = p^{1/(\alpha-1)}$. Thanks to Remark 4.5 (take for instance, $D \equiv -1$), we have that $H_{\text{in}} = H$ inside the region $A = \mathbb{R}^+$, and $H_{\text{in}} = 0$ on the boundary of the region $A = \mathbb{R}^+$ (in fact, in this case, it is easy to see that $A(0) = \{0\}$ which implies $H_{\text{in}} = 0$ on $\partial A$).

It is natural for our problem to look for solutions with positive derivative. In this case, the HJB equation for the unconstrained problem becomes

$$
\rho V(x) - axDV(x) + \frac{1 - \alpha}{\alpha} [DV(x)]^{\alpha/(\alpha-1)} = 0,
$$

and the value function satisfies $V(0) = 0$, which in light of the above remarks about the Hamiltonians is a necessary condition for it to solve the equation in the viscosity sense.

It can be easily checked that the value function is a classical solution of the above equation and is given by $V_u(x) = bx^a$ where $b = (1/\alpha)((\rho - \alpha a)/(1 - \alpha))^{\alpha-1}$. In feedback form, the optimal policy is $c^*(t) = b_1x^*(t)$, where $b_1 = ((\rho - \alpha a)/(1 - \alpha))$. The optimal trajectory $x^*$ starting at a given point $x_0$ is given by $x^*(t) = e^{(a-b_1)t}x_0$ so that $c^*(t) = b_1e^{(a-b_1)t}x_0$.

Let us now consider the constrained case. First, we observe that $V_u$ is greater than $D_0$ when

$$
x \geq x^0 = \left[ \frac{D_0}{b} \right]^{1/\alpha} > 0.
$$
Therefore, we have \( E \subset \{ x_0 : V_u(x_0) \geq D_0 \} = [x^0, +\infty) \). We have the following result, depending on the value of \( \rho/\alpha \).

**Proposition 6.1.**

(i) If \( \alpha \geq \rho \) (which is equivalent to \( \alpha \geq b_1 \)), we have that \( E = [x^0, +\infty) \) and \( V_u = V \) in \( E \) (otherwise, we have \( V = -\infty \)). The optimal policy for the unconstrained problem is also optimal for the constrained one.

(ii) If \( \rho > \alpha \) (which is equivalent to \( \alpha < b_1 \)) then the value function \( V \) is smaller than \( V_u \). In this case, \( E = [\tilde{x}, +\infty) \), where

\[
\tilde{x} = \frac{(\rho \alpha D_0)^{1/\alpha}}{\alpha} > x^0
\]

and there exists a unique optimal policy given by \( \tilde{c}^*(t) = c(x_0)e^{(\alpha - b_1)t} \) (where \( c(x_0) < b_1 x_0 \)) until the point \( x = \tilde{x} \) is reached and then by \( \tilde{c}^*(t) = (\rho \alpha D_0)^{1/\alpha} \).

**Proof.** If \( \alpha \geq \rho \), then the optimal policy for the unconstrained problem is admissible for the constrained one for every initial datum \( x_0 \geq x^0 \) since

\[
\int_0^{\infty} e^{-\rho(s-t)} \frac{[e^s(x)]^\alpha}{\alpha} \, ds = \frac{b_1^\alpha x_0^\alpha e^{\alpha(a-b_1)t}}{\alpha(\rho - \alpha(a-b_1))} \geq \frac{b_1^\alpha x_0^\alpha}{\alpha(\rho - \alpha(a-b_1))}
\]

\[
= \frac{b_1^\alpha - 1}{\alpha} x_0^\alpha = V_u(x_0) \geq D_0.
\]

If \( \rho > \alpha \), then it can easily be shown by looking at the above calculations that the optimal policy for the unconstrained problem \( c(t) = b_1 x(t) \) violates the constraint Eq. (28) for large \( t > 0 \), so that the value function \( V \) is smaller than \( V_u \). From Proposition 5.4, it follows that \( E \) is closed (Assumption 2.3(ii) is satisfied by simply taking the constant control \( c \equiv \alpha y \) at every starting point \( y \in E \)). From Proposition 5.6, it follows that \( E \) is convex and that if \( x \in E \), then also any \( y \geq x \) belongs to \( E \). This means that \( E \) is either empty or \( E = [\tilde{x}, +\infty) \) for some \( \tilde{x} \in (0, +\infty) \). Now, it is clear that the point \( (\rho \alpha D_0)^{1/\alpha} / \alpha \) belongs to \( E \) since for this point, the constant control \( c \equiv (\rho \alpha D_0)^{1/\alpha} \) satisfies the incentive constraint. Indeed, for every \( t \geq 0 \), we have

\[
\int_0^{\infty} e^{-\rho(s-t)} \frac{\rho \alpha D_0}{\alpha} \, ds = D_0.
\]

This fact implies that \( E \) is not empty and that \( \tilde{x} \in [x^0, (\rho \alpha D_0)^{1/\alpha} / \alpha] \). To determine \( \tilde{x} \), we use the maximum principle for state constraints optimal control problems. From Theorem 5.1 we know that the problem

\[
\max J(c) = \max \int_0^{\infty} e^{-\rho s} \frac{c(t)^\alpha}{\alpha} \, ds, \quad \dot{x}(t) = ax(t) - c(t), \quad x(0) = x_0 \int_0^{\infty} e^{-\rho(s-t)} \frac{[e^s(x)]^\alpha}{\alpha} \, ds \geq D_0 \quad \forall t > 0 \quad (29)
\]

corresponds to the following state constraints control problem in the half line \( E = [\tilde{x}, +\infty) \):
\[
\max J(c) = \int_{0}^{+\infty} e^{-\rho t} \frac{c(t)^{\alpha}}{\alpha} \, dt
\]
\[
\dot{x}(t) = ax(t) - c(t), \quad x(0) = x_0, \quad x(t) \geq \bar{x}.
\]

It can be shown that this problem has a unique solution by applying a slight modification of the arguments used to prove Theorem 5.10 (we omit the technicalities here) and Remark 5.11. However, the existence of an optimal strategy can also be proved by checking the sufficient conditions for optimality (see, e.g. Hartl et al., 1995). The current value Hamiltonian and the Lagrangean are, respectively
\[
F(x, c, p) = \frac{c^\alpha}{\alpha} + p(ax - c), \quad L(x, c, p; \mu) = F(x, c, p) + \mu(x - \bar{x}),
\]
the maximum value Hamiltonian is
\[
H(x, p) = \alpha xp + \frac{1 - \alpha}{\alpha} p^{a/(\alpha-1)}
\]

A sufficient condition for \((x, c)\) to be an optimal pair starting at a given point \(x_0\) is that there exists a continuous, piecewise differentiable function \(p: \mathbb{R}^+ \mapsto \mathbb{R}^+\), and a piecewise continuous function \(\mu: \mathbb{R}^+ \mapsto \mathbb{R}^+\) such that
\[
\dot{p}(t) = (\rho - a) p(t) - \mu(t),
\]
\[
\dot{x}(t) = ax(t) - c(t), \quad x(0) = x_0, \quad x(t) \geq \bar{x}, \quad \mu(t) \geq 0, \quad \mu(t)(x(t) - \bar{x}) = 0,
\]
with the transversality condition \(\lim_{t \to +\infty} e^{-\rho t} p(t)x(t) = 0\), where
\[
c(t) = p(t)^{1/(\alpha-1)},
\]
see Hartl et al., 1995.

Assume by contradiction that \(\bar{x} < (\rho a D_0)^{1/\alpha}/a\). Setting the starting point \(x_0 = \bar{x}\), it is easy to check that the control \(c \equiv a\bar{x}\) satisfies the above conditions if we have \(p(t) \equiv (a\bar{x})^{\alpha-1}\) and \(\mu(t) \equiv (\rho - a) p(t) > 0\). This control is then optimal for the state-constrained problem in \(E\). However, from Remark 5.3, such a control is also optimal for the incentive constrained problem. Since it gives a value of the functional strictly less than \(D_0\), we get a contradiction. Therefore, \(\bar{x} = (\rho a D_0)^{1/\alpha}/a\).

The optimal strategy for a given starting point \(x_0 > \bar{x}\) can be found implicitly by solving the above system by straightforward arguments that we outline below. From the equation for the co-state \(p\) we get that, before touching the boundary, we have \(p(t) = p(0)e^{(\rho - a)t}\). It then follows that, before touching the boundary, the optimal control strategy has the form \(\tilde{c}(t) = c(x_0)e^{(a-b_1)t}\) where \(c(x_0) = p(0)^{1/(\alpha-1)}\). The optimal state trajectory is given by
\[
\tilde{x}(t) = \left[ x_0 - \frac{c(x_0)}{b_1} \right] e^{at} + \frac{c(x_0)}{b_1} e^{(a-b_1)t}.
\]
It can be shown that the state–co-state pair \((x, p)\) satisfies the transversality condition and the continuity of \(p\) only when \(c(x_0) < b_1 x_0\) and
\[
c(x_0)[b_1 x_0 - c(x_0)]^{1+b_1/\alpha} = a\bar{x}[(b_1 - a)\bar{x}]^{-1+b_1/\alpha}
\]
(it is enough to impose that $x$ decreases and that at the time $\tilde{t}$ when $x(\tilde{t}) = \bar{x}$, $p$ is continuous). From the uniqueness, it follows that this is the optimal trajectory.

The incentive compatibility constraint affects the agent’s saving process. If the interest rate is higher than the discount rate, then the first best solution satisfies the incentive compatibility constraint, but a large initial stock of capital is needed to have a solution. So, taking into account the fact that the consumer can go abroad in the presence of a fixed cost, then the second best solution gives the first best solution if the interest rate is higher than the discount rate provided that the initial stock of capital is large enough, otherwise, a solution does not exist. In this case, the economy is so rewarding that the agents do not want to go away. If the discount rate is higher than the interest rate, then the second best solution foresees a slower rate of consumption than the first best solution and the constrained value function is smaller than the unconstrained value function. Moreover, a solution for the constrained problem exists provided that the initial stock of capital is larger than a minimal stock of capital, which is higher than the one needed in the first case. If this does not happen, then there is no second best solution.

The HJB equation for the constrained problem is

$$\rho V(x) - H(x, DV(x)) = 0,$$

where, similarly, to the unconstrained case, we have that $H_{in} = H$ inside the region $E$ and $H_{in} = \rho D_0$ on $\partial E$, see Remark 4.5. The value function satisfies $V(\bar{x}) = D_0$ which again is necessary for $V$ to be a viscosity supersolution. The equation is therefore the same as the one for the unconstrained case but it is defined on a different set. However, if $\rho \leq a$, we have $V_1 = V$, whereas for $\rho > a$, we have $V_1 > V$.

**Remark 6.2.** The above HJB equation for the constrained problem does not have, in general, a unique solution (even within the class of classical ones). Indeed, if we allow for linearly growing solutions, then in the case $\rho = a = 1$, $\alpha = 1/2$ and $D_0 = 2$, the HJB equation becomes (since $x^0 = 1$ here)

$$V(x) - xDV(x) - \frac{1}{DV(x)} = 0$$

and $V(1) = 2$. It is easy to check that the functions $V_1(x) = 2x^{1/2}$ and $V_2(x) = x + 1$ are both solutions of Eq. (31). The function $V_1$ is the value function. See, e.g. Soravia, 1997a,b) for an analysis of the non-uniqueness of solutions of Hamilton–Jacobi equations arising in state constraints optimal control problems.

Despite the lack of uniqueness of solutions of the HJB, Eq. (30), we have that the value function is always a viscosity solution of (30) and that it is characterized as in Proposition 4.3. Moreover, by Remark 4.5, $V$ is continuous. The sufficient conditions for the optimal control allow us, in some cases, to compute the value function.
6.2. Optimal investment with a fixed cost

Let us consider the classical optimal investment problem with quadratic adjustment costs and a linear technology:

$$
\max J(k_0; u) = \max \int_0^{+\infty} e^{-\rho t} \left[ ak(t) - bu(s) - \frac{c}{2} u^2(t) \right] dt,
$$

$$
\dot{k}(t) = u(t) - \mu k(t), \quad k(0) = k_0,
$$

where $a > 0$, $b > 0$, $c > 0$, subject to the usual constraint $k \geq 0$ and to the incentive constraint

$$
\int_t^{+\infty} e^{-\rho(s-t)} \left[ ak(s) - bu(s) - \frac{c}{2} u^2(s) \right] ds \geq \tilde{D} \quad \forall t > 0. \quad (32)
$$

$u$ denotes investments and $k$ is the stock of capital. The constant $\tilde{D} > 0$ represents a fixed cost to run the firm.

Set $\tilde{\alpha} = a/(\rho + \mu)$, the expected return from a unit of capital. Assuming that $\tilde{\alpha} \geq b$ (which means that investments are profitable) and choosing measurable control strategies $u$ such that $t \to e^{-\rho(t)} u^2(t)$ are square integrable, we get that the optimal control-state trajectory for the unconstrained problem is

$$
u^*(t) \equiv \frac{1}{c} [\tilde{\alpha} - b], \quad k^*(t) = \frac{u^*}{\mu} + e^{-\mu t} \left[ k_0 - \frac{u^*}{\mu} \right]. \quad (33)
$$

and the unconstrained value function is

$$
V_u(k_0) = \frac{ak_0}{\rho + \mu} + \frac{1}{2c\rho} [\tilde{\alpha} - b]^2.
$$

Recalling that the current value Hamiltonian is defined as

$$
F_0(k, p, u) = (-\mu k + u) p + ak - bu - \frac{c}{2} u^2
$$

$$
= [-\mu kp + ak] + \left[ up - bu - \frac{c}{2} u^2 \right] \overset{\text{def}}{=} F_{01}(k, p) + F_{02}(p; u)
$$

and the maximum value Hamiltonian as

$$
H_0(k, p) = \sup_{u \in \mathbb{R}} F_0(k, p; u) = [-\mu kp + ak] + \left[ \frac{(p - b)^2}{2c} \right] \overset{\text{def}}{=} H_{01}(k, p) + H_{02}(p),
$$

where the maximum point is reached at $u = (p - b)/c$, we observe that $V_u$ can be written as

$$
V_u(k_0) = \tilde{\alpha} k_0 + \frac{1}{\rho} H_{02}(\tilde{\alpha}).
$$

$V_u$ is a viscosity solution (here, also classical) of the HJB equation

$$
\rho V(k) = -\mu kDV(k) + ak + H_{02}(DV(k)); \quad k \geq 0.
$$
The set \( \{ k : V_u(k) \geq \tilde{D} \} \) surely contains the admissible region \( E \) for the constrained problem. The set is given by
\[
[\bar{k}, +\infty), \quad \text{where } \bar{k} = \max \left\{ 0, \frac{1}{\tilde{a}} \left[ \tilde{D} - \frac{1}{\rho} H_{02}(\tilde{a}) \right] \right\}.
\]

We now describe the behavior of the optimal trajectories for the constrained problem. We will not discuss the issue of the HJB equation for the constrained problem, mentioning only that all of the results of Sections 4 and 5.3 can be applied.

**Proposition 6.3.** Depending on the value of \( \bar{D} \), we have

(i) If
\[
\tilde{D} \leq \frac{1}{\rho} H_{02}(\tilde{a})
\]
then for every \( k_0 \geq 0 \), the optimal strategy \( u^* \) for the unconstrained problem is still admissible and optimal for the constrained one. Therefore, \( E = A = [0, +\infty) \) and \( V_u = V \).

(ii) If
\[
\frac{1}{\rho} H_{02}(\tilde{a}) < \tilde{D} \leq \frac{\tilde{a} u^*}{\mu} + \frac{1}{\rho} H_{02}(\tilde{a})
\]
then for every \( k_0 > 0 \), the optimal strategy \( u^* \) for the unconstrained problem is still admissible and optimal for the constrained one. For \( k_0 < \bar{k} \), we have \( V_u(k_0) < \tilde{D} \) and \( V(k_0) = -\infty \). Hence, \( E = [\bar{k}, +\infty) \subset A \) and \( V_u = V \) on \( E \).

(iii) If
\[
\frac{\tilde{a} u^*}{\mu} + \frac{1}{\rho} H_{02}(\tilde{a}) < \tilde{D} \leq \frac{\tilde{a} u^*}{\mu} + \frac{1}{\rho} H_{02}(\tilde{a}) + \frac{\tilde{a}^2 \rho}{2 c \mu^2}
\]
then the optimal policy for the unconstrained problem is no more admissible for the constrained one, no matter what the starting point \( k_0 \) is. In this case, \( E = [\bar{k}, +\infty) \) where \( \bar{k} \) increases with \( \tilde{D}, \tilde{k} > \bar{k} > u^*/\mu \) (the equality holds when \( \tilde{D} \rightarrow (\tilde{a} u^*/\mu) + H_{02}(\tilde{a})/\rho \), \( \tilde{k} \leq (u^*/\mu) + \tilde{a} \rho/(c \mu^2) \) (the equality holds when \( \tilde{D} = (\tilde{a} u^*/\mu) + [H_{02}(\tilde{a})/\rho] + \tilde{a}^2 \rho/(2 c \mu^2) \)). Moreover, \( \bar{k} \) is the smallest solution of the equation
\[
\left[ \tilde{a} + \frac{c \mu u^*}{\rho} \right] - H_{02}(\mu \tilde{k}) = \tilde{D}.
\]

The optimal state trajectory is decreasing till it hits the point \( \bar{k} \) (at a point, say \( t^* \)) and then it is constant. It is always greater (with its derivative, too) than the optimal trajectory for the unconstrained problem. The optimal strategy \( \tilde{u}^* \) for the constrained problem is greater than the one for the unconstrained problem since we have, for \( t \leq t^* \)
\[
\tilde{u}^*(t) = u^* + [c(k_0) - \tilde{a}/c] e^{(\rho+\mu)t}
\]
for a given \( c(k_0) > \tilde{a}/c \). For \( t \geq t^* \) of course, we have \( \tilde{u}^*(t) \equiv \mu \bar{k} > u^* \).
(iv) If 
\[ \bar{D} > \frac{\tilde{a}u^*}{\mu} + \frac{1}{\rho} H_{02}(\tilde{a}) + \frac{\tilde{a}^2 \rho}{2c\mu^2} \]
then \( E \) is empty and there is no solution for the constrained problem.

**Sketch of Proof.** We only sketch the proof since the main arguments are similar to the ones used in the Proof of Proposition 6.1. Setting for \( t \geq 0 \)

\[ J(t, k_0; u) = \int_t^{+\infty} e^{-\rho(s-t)} \left[ ak(s) - bu(s) - \frac{c}{2} u^2(s) \right] ds, \]

by easy calculations we obtain

\[ J(t, k_0; u) = \tilde{a}k(t) + \int_t^{+\infty} e^{-\rho(s-t)} \left[ (\tilde{a} - b)u(s) - \frac{c}{2} u^2(s) \right] ds \]

\[ = \tilde{a}k(t) + \int_t^{+\infty} e^{-\rho(s-t)} F_{02}(\tilde{a}, u(s))ds \]

so that

\[ J(t, k_0; u^*) = \frac{\tilde{a}u^*}{\mu} + \frac{1}{\rho} H_{02}(\tilde{a}) + e^{-\mu t} \tilde{a} \left[ k_0 - \frac{u^*}{\mu} \right] \geq \frac{1}{\rho} H_{02} \left( \frac{a}{\rho + \mu} \right) \forall t \geq 0 \]

and

\[ \lim_{t \to +\infty} J(t, k_0; u^*) = \frac{au^*}{\mu(\rho + \mu)} + \frac{1}{\rho} H_{02} \left( \frac{a}{\rho + \mu} \right). \]

Then, by imposing the incentive constraint Eq. (32) we easily obtain (i) and (ii) and that, for

\[ \tilde{D} > \frac{\tilde{a}u^*}{\mu} + \frac{1}{\rho} H_{02}(\tilde{a}), \]

the strategy \( u^* \) is not admissible for the constrained problem, no matter what the starting point \( k_0 \) is. We now analyze the form of the region \( E \) and the optimal state-control trajectories for the constrained problem by the techniques used in the Proof of Proposition 6.1. We first apply Proposition 5.4 (Assumption 2.3(ii) is satisfied here by simply taking the constant control \( c \equiv -\mu k \) at every starting point \( k \in E \)) and Proposition 5.6 to get that the region \( E \) is a half-line contained in \([\hat{k}, +\infty)\) or is empty.

Assume that \( E \) is non-empty. Then, it must be \( E = [\hat{k}, +\infty) \) for some \( \hat{k} \geq \hat{k} \). By Theorem 5.1, we know that the constrained problem is equivalent to the state constraints problem in the region \( E \). Then we can use (as in Proposition 6.1) the sufficient conditions for optimality for the state constraints problem in the region \( E \) (see Hartl et al., 1995) to get that at \( k_0 = \hat{k} \), the constant control strategy \( \tilde{u}^* \equiv \mu \hat{k} \) is optimal for the constrained problem. Still arguing as in Proposition 6.1, we then get that \( \hat{k} \) must be characterized by the property that

\[ J(\hat{k}; \tilde{u}^*) = \hat{D}. \]
By easy calculations, we can see that \( J(\bar{k}; \bar{a}^*) \) is a quadratic concave function of \( \bar{k} \) and has the maximum at the point \( (\bar{u}^*/\mu) + \bar{a}^*/(c\mu^2) \) whose value is \( (\bar{u}^*/\mu) + [H_{02}(\bar{a})/\rho] + \bar{a}^2\rho/(2c\mu^2). \) This means that for \( \bar{D} \) strictly greater than this value, we get a contradiction and so we must have \( \bar{E} = \emptyset \) (case iv). For \( \bar{D} \) less than or equal to this value, we compute \( \bar{k} \) by solving Eq. (34). To find the optimal policy in the case (iii) and to show that \( c(k_0) > 0 \) we use, again as in Proposition 6.1, the sufficient conditions for optimality.

As the fixed cost goes up, we observe an interesting scenario.

- If it is low enough, then neither the optimal policy nor the state region allowing existence of the optimal solution change with respect to the unconstrained case.
- If the fixed cost increases, then the optimal policy is the first best solution but a large enough stock of capital is required to run the firm. The fixed cost does not affect the optimal investment policy, but the state region allowing existence of the optimal solution becomes smaller.
- If the fixed cost is furthermore increased, then the state region allowing existence of the optimal solution is furthermore restricted and a second best optimal control strategy is obtained. The investment rate is higher than the first best solution. In this parameters’ region, the fixed cost affects both the optimal investment strategy and the state region.
- Finally, if the fixed cost goes beyond a certain level, then there is no possibility to recover it, no matter what the initial stock of capital is.

Summing up, the incentive compatibility constraint has two effects with respect to the unconstrained problem: it restricts the state region for which a solution exists, it induces a higher rate of capital accumulation and a smaller value function. The optimal policy foresees a stationary level of the state variable when the incentive constraint becomes binding.

7. Conclusions

In this paper, we have analyzed dynamic incentive compatibility constrained problems in continuous time. The incentive constraint is a constraint on the continuation value of the payoff function. More precisely, at every time the residual payoff is supposed to be greater than or equal to a certain function of the state and/or of the control. We have characterized the value function associated with the constrained problem by proving that the dynamic programming principle holds and that it is a viscosity solution of the HJB equation. Restricting our attention to an incentive compatibility constraint, which only depends on the value of the state, we have shown the equivalence of the constrained problem with a state-constrained problem in an endogenous region. This equivalence is useful to define the optimal strategy by means of the Pontryagin maximum principle.

Two simple economic problems have been analyzed where the incentive compatibility constraint is given by a positive constant. We have shown that the constrained problem coincides with the unconstrained problem only in some cases. In general, as the constraint becomes more binding, we have three effects: the state region allowing existence for the constrained problem shrinks, the rate of capital accumulation becomes higher than the first best rate, and the value function becomes smaller than the one obtained in the unconstrained problem.
Acknowledgements

We thank Aldo Rustichini for suggesting the problem, Piermarco Cannarsa for helpful remarks, and the participants of the seminars in AMASES congress (Roma, Italy), IAC-CNR (Roma, Italy), Georgia Tech (Atlanta, USA), II SCE Workshop in Computing in Economics and Finance (Cambridge, UK), Università di Roma “La Sapienza” (Italy), Università di Padova. The usual disclaimers apply. Fausto Gozzi and Andrzej were partially supported by NSF grant DMS 97-06760, Emilio Barucci was partially supported by a CNR grant.

Appendix A

Proof of Lemma 3.2. Given a $c \in \mathcal{C}(x_0)$, we have to prove that

$$\int_t^{+\infty} e^{-\rho s} f_0(x(s; x(T; x_0, c), c_T(s))) ds \geq e^{-\rho t} D(x(t; x(T; x_0, c), c_T(t))).$$

where $c_T(r) = c(T + r)$ for $r \geq 0$ and $x(r; x(T; x_0, c), c_T) = x(r + T; x_0, c)$. Therefore, the above inequality is equivalent to

$$\int_t^{+\infty} e^{-\rho s} f_0(x(s + T; x_0, c), c(T + s))) ds \geq e^{-\rho t} D(x(t + T; x_0, c), c(T + t)), \quad \text{a.e. } t \geq 0$$

and, by the change of variable $\sigma = s + T$,

$$\int_{t+T}^{+\infty} e^{-\rho \sigma} f_0(x(\sigma; x_0, c), c(\sigma)) d\sigma \geq e^{-\rho(t+T)} D(x(t + T; x_0, c), c(T + t)), \quad \text{a.e. } t \geq 0$$

which follows from the fact that $c \in \mathcal{C}(x_0)$. This implies that $c_T \in \mathcal{C}(x(T; x_0, c))$. \qed

Proof of Lemma 3.3. Given a $c \in \mathcal{C}(x_0)$ and the control $c_1$ defined in Eq. (18), we want to prove that

$$\int_t^{+\infty} e^{-\rho s} f_0(x(s; x_0, c_1), c_1(s)) ds \geq e^{-\rho t} D(x(t; x_0, c_1), c_1(t)) \quad \text{a.e. } t \geq 0.$$

Let first $t \geq T$. Since $x(r; x_0, c_1) = x(r - T; x(T; x_0, c), \tilde{c}_T)$ for $r \geq T$, then for every $t > T$, the latter inequality is equivalent to

$$\int_t^{+\infty} e^{-\rho s} f_0(x(s - T; x(T; x_0, c), \tilde{c}_T), \tilde{c}_T(s - T)) ds \geq e^{-\rho t} D(x(t - T; x(T; x_0, c), \tilde{c}_T), \tilde{c}_T(t - T))$$

and, by the change of variable $\sigma = s - T$, to
\[ \int_{-T}^{+\infty} e^{-\rho \sigma} f_0(x(\sigma; x(T; x_0, c), \tilde{c}_T), \tilde{c}_T(\sigma)) d\sigma \geq e^{-\rho(T-T)} D(x(t - T; x(T; x_0, c), \tilde{c}_T), \tilde{c}_T(t - T)) \]

which follows from the fact that \( \tilde{c}_T \in \mathcal{C}(x(T; x, c)) \). Therefore, for every \( t \geq T \), the constraint Eq. (9) is satisfied.

Let now \( t \in [0, T) \). In this case

\[ \int_{t}^{+\infty} e^{-\rho s} f_0(x(s; x_0, c_1), c_1(s)) ds = \int_{t}^{+\infty} e^{-\rho s} f_0(x(s; x_0, c), c(s)) ds + \left[ \int_{t}^{+\infty} e^{-\rho s} f_0(x(s; x_0, c_1), c_1(s)) ds - \int_{t}^{+\infty} e^{-\rho s} f_0(x(s; x_0, c), c(s)) ds \right]. \tag{35} \]

Let \( J(x(T; x_0, c); \tilde{c}_T) \geq J(x(T; x_0, c); c) \). Then it is easy to show that

\[ \int_{T}^{+\infty} e^{-\rho s} f_0(x(s; x_0, c_1), c_1(s)) ds - \int_{T}^{+\infty} e^{-\rho s} f_0(x(s; x_0, c), c(s)) ds \geq 0. \tag{36} \]

Following the argument we have employed in the first part of the proof, we have that

\[ \int_{T}^{+\infty} e^{-\rho s} f_0(x(s; x_0, c_1), c_1(s)) ds = \int_{T}^{+\infty} e^{-\rho s} f_0(x(s - T; x(T; x_0, c), \tilde{c}_T), \tilde{c}_T(s - T)) ds = e^{-\rho T} \int_{0}^{+\infty} e^{-\rho \sigma} f_0(x(\sigma; x(T; x_0, c), \tilde{c}_T), \tilde{c}_T(\sigma)) d\sigma = e^{-\rho T} J(x(T; x_0, c); \tilde{c}_T), \]

where \( \sigma = s - T \). Similarly

\[ \int_{T}^{+\infty} e^{-\rho s} f_0(x(s; x_0, c), c(s)) ds = e^{-\rho T} J(x(T; x_0, c); c). \]

Therefore the inequality in Eq. (36) is established. The claim now follows from the fact that \( c \in \mathcal{C}(x_0) \), which implies that

\[ \int_{t}^{+\infty} e^{-\rho s} f_0(x(s; x_0, c), c(s)) ds \geq e^{-\rho T} D(x(t; x_0, c), c(t)) \quad \text{a.e. } t \geq 0. \]

Since \( c(t) = c_1(t) \) for \( t \in [0, T) \) and \( c \in \mathcal{C}(x_0) \), it follows from Eqs. (35) and (36) that
\[
\int_{t}^{+\infty} e^{-\rho s} f_0(x; x, c), \ c_1(s)) ds \\
\geq \int_{t}^{+\infty} e^{-\rho s} f_0(x(s; x, c), c(s)) ds \geq e^{-\rho t} D(x(t; x_0, c), c_1(t)).
\]

Therefore, for every \( t \in [0, T) \), the constraint Eq. (9) is satisfied by the control \( c_1 \). \( \square \)

**Proof of Proposition 4.3.** The proof generalizes, improves, and simplifies the proof of the first part of Theorem 2.1 in (Świȩch, 1996). Let \( c \in C(\bar{x}) \), \( T > 0 \), and let \( R \) be such that \( \|x(\cdot, \bar{x}, c)\|L^\infty([0, T]) \leq R \). Denote \( \|u\|L^\infty(\bar{E} \cap B(0, R+1)) \), Let

\[
u_{\epsilon}(x) = \inf_{y \in \bar{E} \cap B(0, R+1)} \left\{ u(y) + \frac{|y - x|^2}{2\epsilon} \right\}
\]

be the inf-convolution of \( u \). It is Lipschitz continuous and semi-concave on \( \bar{E} \cap B(0, R) \), and \( u_{\epsilon} \not\nearrow u \) pointwise. Let \( E_{\epsilon} = \{ x \in \bar{E} \cap B(0, R) : \text{dist}(x, \partial E) > 2\sqrt{K\epsilon} \} \). Denote by \( x^+ \) a point such that

\[
u_{\epsilon}(x) = u(x^+) + \frac{|x^+ - x|^2}{2\epsilon}.
\]

The point \( x^+ \) may not be unique. If \( \epsilon < \epsilon_0 \), \( 2\sqrt{K\epsilon} < 1 \), and \( x \in E_{\epsilon_0} \), then \( x^+ \in \text{int} \bar{E} \). We will be denoting by \( D^+u_{\epsilon}(x) \) (respectively, \( D^-u_{\epsilon}(x) \)) the generalized superdifferential (respectively, subdifferential) of \( u_{\epsilon} \) at \( x \) (see Crandall et al., 1992). We notice that \( D^+u_{\epsilon}(x) \) is always non-empty since \( u_{\epsilon} \) is semi-concave. It is rather standard to notice that if \( p \in D^-u_{\epsilon}(x) \) then \( p \in D^-u(x^+) \), \( p = (x^+ - x)/\epsilon \) and thus

\[
0 \leq \rho u(x^+) - H_u(x^+, p) = \rho u(x^+) - H \left( x^+, \frac{x^+ - x}{\epsilon} \right).
\]

Therefore

\[
0 \leq \rho u_{\epsilon}(x) - H(x, p) + \rho \frac{|x^+ - x|^2}{2\epsilon} + L \frac{|x^+ - x|^2}{\epsilon}.
\]

Denote

\[
\omega(x, \epsilon) = \sup_{x^+} \left\{ \left( \frac{\rho}{2} + L \right) \frac{|x^+ - x|^2}{\epsilon} \right\}.
\]

It is easy to check that \( \lim_{\epsilon \to 0} \omega(x, \epsilon) = 0 \) (see Crandall et al., 1992) and that \( \omega(\cdot, \epsilon) \) is upper-semicontinuous. Therefore, \( u_{\epsilon} \) satisfies

\[
\rho u_{\epsilon} - H(x, Du_{\epsilon}) \geq -\omega(x, \epsilon).
\]

Let now \( p \in D^+u_{\epsilon}(x) \). Then

\[
p = \lim_{n \to \infty} \sum_{i=1}^{n} \lambda_i^n p_i^n, \quad \text{where} \sum_{i=1}^{n} \lambda_i^n = 1, \ p_i^n = Du_{\epsilon}(x_i^n), \ |x_i^n - x| \leq \frac{1}{n}.
\]
Using the convexity of $H$, and lower-semicontinuity of $H$ and $-\omega(\cdot, \epsilon)$ we now have

$$
\rho u(\epsilon) - H(x, p) \geq \rho u(\epsilon) - H \left( x, \sum_{i=1}^{n} \lambda_i^n p_i^n \right) - \sigma_1(n)
$$

$$
\geq \sum_{i=1}^{n} \lambda_i^n (\rho u(\epsilon) - H(x^n_i, p_i^n)) - \sigma_2(\epsilon, n)
$$

$$
\geq -\sum_{i=1}^{n} \lambda_i^n \omega(x^n_i, \epsilon) - \sigma_2(\epsilon, n)
$$

$$
\geq -\omega(x, \epsilon) - \sigma_3(\epsilon, n),
$$

and letting $n \to \infty$, we obtain

$$
\rho u(\epsilon) - H(x, p) \geq -\omega(x, \epsilon) \text{ for } x \in E_{e_0}.
$$

(37)

Let

$$
\tau_{e_0} = \min\{ \inf \{t : x(t; \bar{x}, c) \notin E_{e_0}\}, T\}.
$$

It follows from Eq. (37) that

$$
\rho u(\epsilon)(x(s)) - \langle f(x(s; \bar{x}, c), c(s)), p_s \rangle - f_0(x(s; \bar{x}, c), c(s)) \geq -\sigma(x(s; \bar{x}, c), \cdot),
$$

(38)

for $0 \leq s \leq \tau_{e_0}$, $p_s \in D^+ u(\epsilon)(x(s; \bar{x}, c))$, where $\sigma(x, \epsilon) \to 0$ as $\epsilon \to 0$. Since $\bar{x}$ and $c$ are fixed to simplify notation we will write $x(s)$ for $x(s; \bar{x}, c)$. Denote $\psi(s) = u(\epsilon)(x(s))$. The function $\psi$ is in $W^{1,1}(0, \tau_{e_0})$. To see this, one notices that if $u^{\delta}$ are smooth mollifications of $u$ then $u^{\delta}(x(s))$ converges weakly in $W^{1,1}(0, \tau_{e_0})$ and also converges in $C[0, \tau_{e_0}]$ to $\psi(s)$. Let us now choose a point $s_0$ such that $\psi'(s_0)$ and $x'(s_0)$ exist. The semiconcavity of $u$ gives us

$$
u(\epsilon)(x) \leq u(\epsilon)(x(s_0)) + \langle p_{s_0}, x - x(s_0) \rangle + \frac{|x - x(s_0)|^2}{2\epsilon}
$$

Therefore

$$
\frac{\psi(s) - \psi(s_0)}{s - s_0} \leq (\geq) \left\{ p_s, \frac{x(s) - x(s_0)}{s - s_0} \right\} + \frac{|x(s) - x(s_0)|^2}{2\epsilon (s - s_0)}
$$

if $s > s_0$ (respectively, $s < s_0$). This, together with the fact that $\psi'(s_0)$ and $x'(s_0)$ exist, yield that $\psi'(s_0) = (p_{s_0}, x'(s_0))$ for almost all $s_0 \in (0, \tau_{e_0})$. Hence, we can integrate Eq. (38) to obtain

$$
u(\epsilon)(\bar{x}) \geq \int_0^{\tau_{e_0}} e^{-\rho s} f_0(x(s; \bar{x}, c), c(s)) ds + e^{-\rho \tau_{e_0}} u(\epsilon)(x(\tau_{e_0}; \bar{x}, c))
$$

$$
- \int_0^{\tau_{e_0}} \sigma(x(\tau_{e_0}; \bar{x}, c), c(\tau_{e_0}), \epsilon) ds.
$$
Letting $\epsilon \to 0$ and using the Lebesgue dominated convergence theorem yield
\[
\begin{align*}
u(\tilde{x}) &\geq \int_{\epsilon_0}^{\epsilon_0} e^{-\rho s} f_0(x(s; \tilde{x}, c), c(s)) ds + e^{-\rho t_0} u(x(t_0; \tilde{x}, c)).
\end{align*}
\]
We now let $\epsilon_0 \to 0$ to obtain
\[
\begin{align*}
u(\tilde{x}) &\geq \int_0^\tau e^{-\rho s} f_0(x(s; \tilde{x}, c), c(s)) ds + e^{-\rho \tau} u(x(\tau; \tilde{x}, c)),
\end{align*}
\]
where $\tau = \min\{\inf\{t : x(t; \tilde{x}, c) \in \partial E\}, T\}$. If $\tau = T$, we have
\[
\begin{align*}
u(\tilde{x}) &\geq \int_0^T e^{-\rho s} f_0(x(s; \tilde{x}, c), c(s)) ds + e^{-\rho T} u(x(T; \tilde{x}, c)).
\end{align*}
\] (39)
If $x(\tau; \tilde{x}, c) \in \partial E$, then $u(x(\tau; \tilde{x}, c)) \geq V(x(\tau; \tilde{x}, c))$ and using the dynamic programming principle, we arrive at
\[
\begin{align*}
u(\tilde{x}) &\geq \int_0^T e^{-\rho s} f_0(x(s; \tilde{x}, c), c(s)) ds + e^{-\rho T} V(x(T; \tilde{x}, c)).
\end{align*}
\] (40)
If $\tau = T$ for every $T$, then we let $T \to \infty$ in Eq. (39), otherwise we use Eq. (40). This yields $u \geq V$ in $E$. \hfill \Box

References


