A MODEL FOR THE OPTIMAL ASSET-LIABILITY MANAGEMENT FOR INSURANCE COMPANIES

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This paper is devoted to the formulation of a model for the optimal asset-liability management for insurance companies. We focus on a typical guaranteed investment contract, by which the holder has the right to receive after T years a return that cannot be lower than a minimum predefined rate \( r_g \). We take account of the rules that usually are imposed to insurance companies in the management of this funds as reserves and solvency margin. We formulate the problem as a stochastic optimization problem in a discrete time setting comparing this approach with the so-called hedging approach. The utility function to maximize depends on various parameters including specific goals of the company management.

Some preliminary numerical results are reported to ease the comparison between the two approaches.

Keywords: Portfolio optimization; asset-liability management; transaction costs.

1. Introduction

This paper is devoted to the formulation of a model for the optimal asset-liability management for insurance companies.

Type of contract:
we focus on a typical guaranteed investment contract, by which the holder has the right to receive after T years a return that cannot be lower than a minimum predefined rate \( r_g \).
If the return of the investment made by the company with the money received by the policy-holder is higher than the minimum guaranteed return the holder of the contract shares in the profit for a fixed percentage \(1 - \beta\).

In this kind of contract the holder pays the cost of the minimum guaranteed return (zero-coupon bond) by giving the company a participation \(\beta\) in the fund yield, if the fund performance is better than the minimum guaranteed amount.

Management of the fund:

the company has the right to share in the profit at given times during the lifetime of the contract but must obey some rules, imposed by regulation authorities, on the management of the fund. These rules basically require the company to invest a fixed percentage of the liabilities in non-risky assets in order to guarantee a “solvency” condition at maturity, imposing at the same time a constraint on the percentage of wealth invested in risky assets. If the final value of the fund is less than the minimum guaranteed amount the company has to provide the minimum to policy-holders, thus incurring in potential loss.

The goal:

the company can manage the investment by tuning the weights of the sections in the fund, in order to satisfy the regulatory constraints, to guarantee a solvency condition at maturity and to achieve a given objective, like the maximization of the profit for shareholders.

The profit margins on this kind of policies, that constitute the bulk of liabilities of continental European insurance companies, have been declining in recent years, due to a combination of low inflation and low bond yields, as the profit margins are proportional to the gap between the risk-free rate and the minimum guaranteed return.

As a consequence, the minimum guaranteed return has become particularly valuable to policy-holders whereas it jeopardizes the solvency of the issuing company [11–13].

Life insurance companies have traditionally not paid much attention to the proper valuation of the various elements embedded in their policies (see introduction of [12]) but the reduction in profit margins and the increase of default risk has caused them many troubles since they missed analytical tools for the proper valuation of their obligations.

In the United States a number of companies failed to fulfill their obligations and defaulted [4]. Large financial groups, like the Nissan Mutual Life in Japan, have collapsed [12]. Some insurance companies try to face the rapid decrease of the interest rates towards minimum values with a mixed strategy that increases the bond duration and decreases the minimum guaranteed interest rate.
This policy however can expose the firm to the risk coming from the competition with other products (such as mutual funds) that could provide higher returns [11, 12].

The increasing competition leaves the insurance companies with a difficult alternative: either they accept losing market if they stick to a matched asset-liability policy, or they try to maintain their client base by accepting to risk more.

Here we describe two possible approaches for the companies.

Hedging approach:

A possible approach followed by many insurance companies is to minimize the downside risk embedded in the policy by means of the tools developed in the financial derivatives market. A close examination of a typical guaranteed investment contract reveals an exchange of options between the company and policy-holders where the company is short a floor option on the fund (the minimum guaranteed return) whereas is long a call on the excess return of the fund with respect to the floor (the participation rate), see Fig. 1. Of course such a decomposition is not unique (one could for instance highlight the presence of a put option sold by the company). Moreover some of the contracts contain only European type options, meaning that the options can be exercised only at maturity, whereas others contain also American type options, that can be exercised at any time during the life of the contract. This is the case when the contract allows the holder to surrender (sell-back) the policy before maturity.

From the technical viewpoint it is not easy to follow the hedging approach. In theory one should replicate an option on the underlying fund but this is not a traded security. The price of any option written on the fund it is hard to determine since it depends strongly on the management strategy. Last, but not least, by sticking to the minimum guaranteed path, does not let the management choose the “best” strategy, with respect to a given objective, among all the “admissible” ones.

Optimization approach:

A more general and flexible approach aims to maximize a given objective function among all the admissible strategies. The model is supposed to give the optimal sequence of portfolio adjustments satisfying both regulatory and solvency constraints and leading on to the achievement of a given objective. In this stochastic optimization framework the solvency condition is included as a constraint on the admissible strategies and the function to be maximized takes into account the objective to be achieved.

The rest of the paper is organized as follows: in Sec. 2 we describe the type of contract we consider, in Sec. 3 we give details about the hedging approach and the difficulties it entails whereas in Sec. 4 we present the alternative stochastic optimization approach. Section 5 describes the model. In Sec. 6 we report the
results of some preliminary numerical tests. Section 7 is about conclusions and further developments.

2. Type of Contract

We focus on a standard “European” life insurance contract organized as follows: the policy-holder makes a single-sum deposit \( L_0 \) at time \( t = 0 \). The money is invested in a portfolio of assets, that we call the “fund” hereafter. The fund is divided in “sections” that correspond to the single assets. The company undertakes to pay at maturity \( t = T \) an amount equal to

\[
\pi_T^L = \max(A_T, G_T) \tag{1}
\]

where \( A_T \) is the net-value of the fund at the maturity date \( T \) corresponding to the initial investment \( L_0 \). By net-value, we mean the sum of the actual values at time \( T \) of each asset. The cost of managing the fund is in charge of the company (i.e., the policy holder does not pay a fee for the fund management). \( G_T = L_0 e^{r g T} \) is the minimum guaranteed pay-out at time \( T \). We consider \( r g \) predefined and constant for the entire period \( T \).

The policy-holder finances the cost of the minimum guaranteed return by granting to the insurance company a percentage \( \beta \) on the fund yield if this yield is higher than minimum guaranteed return (cf. Fig. 1). The company withdraws from the fund the amount

\[
p_{t_j} = \beta [A_{t_j} - A_{t_{j-1}}]^+ \tag{2}
\]

where \([x]^+ := \max(x, 0)\) is the positive part of \( x \), \( A_{t_j} \) is the value of the fund at time \( t_j \) which belongs to a set of times \( \{t_1, \ldots, t_p\} \subset [0, T] \). The exact relation among the \( A_{t_j}, A_{t_{j+1}}, \ldots \) is not trivial and we define it in Eq. (22).

This choice of including in the model the withdrawals described by Eq. (2) is due to a specific requirement of the insurance company which has supplied us with its legal procedures and business practices.
Note that these withdrawals can not be considered management fees. They are part of the company policy since the company may invest the money of the withdrawals in other assets not belonging to the fund. Following this approach the company does not take into account \( r_g \) to decide whether a withdrawal is possible. An alternative policy (which is not considered in the present work) is given in Eq. (3):

\[
p_{t_j} = \beta \left[ A_{t_j} - L_0 e^{r_g t_j} \right]^+.
\]  

We focus on single-premium policy, leaving aside demographic risk and surrender options, but our model can be easily extended to include the case of delayed payments, non homogeneous funds and mortality risk. Much work \([12, 1, 11]\) has been recently done about the fair valuation of such products.

3. Hedging Approach

The final payoff for the policy-holder can be split into a series of contingent-claims on the final worth \( A_T \) of the trading account \([11]\). For example one could highlight the presence of a zero-coupon bond plus a call option:

\[
\pi_T^L = G_T + \max(A_T - G_T, 0)
\]  

or the presence of a put option sold by the company that represents the main source of insolvency risk for the firm:

\[
\pi_T^p = A_T + \max(G_T - A_T, 0).
\]

The company has, of course, the opposite position at maturity (cf. Fig. 2):

\[
\pi_T^E = -\max(G_T - A_T, 0)
\]  

although this decomposition alone does not take into account the profit for the company.

As described by Eq. (2), the company may withdraw a percentage of the excess return of the fund. The corresponding net profit is given by the sum of all the withdrawals, \( \{p_{t_j}\} \) plus a return which depends on the withdrawal time and how the money of the withdrawal is invested. For the sake of simplicity we consider that this profit can be described by the following expression:

\[
P_T = \sum_{j=1}^p p_{t_j} e^{f(t_j, T)(T-t_j)}
\]  

where \( f(t_j, T) \) is the forward rate from \( t_j \) to \( T \).

In summary if the final value \( A_T \) of the investment is less than the minimum pay-out \( G_T \), the company has to provide the minimum guaranteed amount with its own capital, thus incurring in potential loss and default risk.

The hedging approach used in insurance companies is based on the replication of the put option sold to policy-holders. This minimizes the downside risk of the
final payoff, i.e. the risk of $A_T$ being less than $G_T$. This replication is usually performed using the standard tools developed for the derivatives market (Black-Scholes analysis and $\Delta$-hedging). However a close examination of this approach reveals several difficulties:

- The claims that appear in this context are contingent on the final worth $A_T$ of the underlying fund, hence they depend on the strategy used to manage the fund. It is not trivial to define a price for this kind of derivatives and moreover every change in the portfolio changes the value of the contingent claims as well. The long-term effect of this kind of nonlinearity is not clear (see also the introduction).
- The fund $(A_t)$ is not a traded security. A way to overcome this difficulty is to apply the hedging to the single assets included in the portfolio. This means to consider the option held by the policy-holder as a portfolio of options.
- The hedging technique may require a continuous adjustment of the portfolio to be fully consistent. Since the transaction costs are in charge of the company, it is desirable to minimize the number of adjustments. This means that another trade-off must be realized.

All these difficulties make clear how the standard hedging technique can be quite difficult to apply. A more general approach called “shortfall risk minimization” [8, 11, 10, 14] consists in accepting some risk lowering on the other side the initial
investment. Again if we require the accepted risk to be small the initial investment can be very high.

However there is a more fundamental motivation to look for a different approach. Besides granting the solvency condition at maturity, an efficient management should tend to achieve one or more objectives: maximization of the profit for the shareholders, control on the fund fluctuations, out-performance of a given benchmark and so on.

By sticking to the minimum guaranteed path, the replication approach does not let the manager choose the “best” strategy among all the “admissible” ones, i.e. among all the strategies that ensure the solvency condition $A_T \geq G_T$.

4. Stochastic Optimization Approach

By now, stochastic optimization has been recognized as an important tool for dealing with multi-period optimization problems.

We are going to develop a new discrete-time model for the optimal management of an insurance asset portfolio. The model is supposed to give the optimal sequence of portfolio adjustments satisfying several constraints leading on to the achievement of a given objective. The main constrain is the solvency condition $A_s \geq G_s$. This should hold $\forall s \in [0, T]$ but in fact the company check for solvency only periodically. So we require that this holds for $s$ belonging to a given set $s_0, \ldots, s_r$.

In a stochastic optimization framework the solvency condition is included as a constraint on the admissible strategies and the cost functional takes into account the objective to be achieved.

The model must consider the actual management and the discrete structure of the drawings from the fund (this is the reason why we decided to start with a discrete-time model). It must take into account a number of constraints, both of endogenous and exogenous nature, embedded in the problem and the effect of transaction costs on the choice of the “best” strategy. Finally the model should be as “parametric” as possible to provide high flexibility: the management should be able to provide a suitable benchmark to the sections of the fund; it should be free to choose the “cost functional” which describes a given objective (maximization of the profit, control on the fund fluctuations, minimization of the shortfall risk) and it should be allowed to cut off the strategies leading on to an unwanted result.

The implementation of stochastic optimization models is the subject of our current research. For what concerns discrete-time models there are two possible approaches: a “static” stochastic optimization, where all the controls at all time-steps are regarded as independent variables and a dynamic-programming algorithm ([2] and [14] for an application to shortfall risk minimization).

We think that the “static” technique can be useful since it may provide a quick and feasible tool to test the reliability of the model, even if it can lead on to a large number of variables when the number of time-steps is high. In our case we were interested in simulations spanning up to ten years, with a possible portfolio-adjustment
every three months. That means about forty time-steps, that is still a treatable amount in this context.

Alternative and more flexible approaches include the discrete-time dynamic programming algorithm and the development of a continuous-time model for the management of the fund using continuous-time stochastic dynamic programming.

In the latter case the optimal investment strategy is provided by the solution of variational inequalities in the portfolio yield function in a “feedback” form, i.e. as a function of the current state of the system. This approach leads to non-linear second order degenerate-parabolic partial differential equations (Hamilton-Jacobi-Bellman equations) that usually do not admit an explicit solution, and a numerical approach to the problem is needed. It is then necessary to study the numerical approximation techniques and to develop an efficient algorithm to solve the HJB equation in high dimensional spaces.

5. The Model

5.1. The evolution of the assets

We consider a guaranteed investment contract defined by the triplet \((r_g, \beta, T)\) and assume that the liabilities cash-flow is just the initial deposit \(L_0\), that means to neglect the case of periodic payments, mortality, surrender-options.

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with \(\mathcal{F}\) a \(\sigma\)-algebra over \(\Omega\) and \(\mathbb{P}\) a probability measure, and let \(W\) a standard \(n + 1\)-dimensional Brownian motion over \((\Omega, \mathcal{F}, \mathbb{P})\) with respect to the filtration \((\mathcal{F}_t)_t\). The fund is composed by \(n + 1\) sections: one non-risky asset with price \(U^{(0)}\) and \(n\) risky assets with prices \(U^{(1)}, \ldots, U^{(n)}\) whose evolution is given by

\[
\begin{align*}
U^{(0)}_{j+1} &= U^{(0)}_j e^{r_j \Delta t_j}, \\
U^{(0)}_0 &\geq 0,
\end{align*}
\]

with \(\Delta t_j := t_{j+1} - t_j\) and

\[
\begin{align*}
U^{(i)}_{j+1} &= U^{(i)}_j e^{\mu^{(i)} \Delta t_j + \sigma^{(i)} \cdot \Delta W_j}, \\
U^{(i)}_0 &\geq 0,
\end{align*}
\]

for \(i = 1, \ldots, n\), \(\sigma^{(i)} := (\sigma^{(i,0)}, \ldots, \sigma^{(i,n)})\), \(\Delta W_j := (\Delta W^{(0)}_j, \ldots, \Delta W^{(n)}_j)\) and the interest rate \(r_j\), is (following [3]) \(\max(0, \hat{r}_j)\) where \(\hat{r}_j\) is the solution of the following differential equation at time \(t_j\),

\[
\begin{align*}
d\hat{r}_t &= \hat{r}_t \mu_t dt + \sigma_t dW^{(0)}_t, \\
\hat{r}_0 &> 0.
\end{align*}
\]

\(^a\)Note that here ten years is the life time of the fund. According to the information provided by the company we dealt with, this time frame it is not necessarily equal to the life time of the insurance policies.
The choice of a particular model for the evolution of the interest rate is only for sake of simplicity, as it does not influence the general setting of the paper. At times, we will indicate the equations above as

\[
\begin{cases}
U_{j+1}^{(i)} = f_i(U_j^{(i)}, \Delta W_j), \\
U_0^{(i)} \geq 0,
\end{cases}
\]

where the function \(f_i : \mathbb{R} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}\) is defined by:

\[
f_i(x, (w_0, \ldots, w_n)) = \begin{cases}
x e^{(r_j - t_j) \Delta t_j} & i = 0, \\
x \exp \left( \mu(i) \Delta t_j + \sum_{k=0}^{n} \sigma^{(i,k)} w_k \right) & i \neq 0,
\end{cases}
\]

for every \(x, w_0, \ldots, w_n \in \mathbb{R}\). Finally we observe that following simple computations the solution \(\hat{r}_t\) of (10) can be written as

\[
\hat{r}_t = \hat{r}_0 e^{\mu r_t} + \sigma_r \int_0^t e^{\mu r(t-s)} dW_s(0)
\]

and since the integrand in the Ito's integral is a deterministic function of time, it follows

\[
r_j \sim \max(0, N(\hat{r}_0 e^{\mu r_j}, \sigma_r^2 \phi_j(\mu_r))),
\]

where the real valued function \(a \mapsto \phi_j(a)\) is defined by

\[
\phi_j(a) = \begin{cases}
e^{2at_j} - 1 & \text{if } a \neq 0, \\
\frac{2a}{t_j} & \text{if } a = 0,
\end{cases}
\]

for every \(j = 1, 2, \ldots\). We use this distributional feature of the interest rate path mainly to simulate it by means of a Gaussian number generator.

The drift and diffusion parameters appearing in the equations above have to be estimated starting from market data.

### 5.2. The evolution of the fund

#### 5.2.1. The transactions between the sections

The company manages the fund by tuning the weights of the sections in the fund by means of transactions occurring at any time between the sections.\(^b\)

Let \(S_j^{(i)}\) be the total amount of money invested in the asset \(U^{(i)}\) at time \(t_j, i = 0, \ldots, n\). We introduce the controls \(C_{j}^{rk} := \text{percentage of the section } S^{(r)} \text{ moved to the section } S^{(k)} \text{ at time } t_j\).

\(^b\)A more general approach sometimes used by the management, is based on the idea of using derivatives to manage the portfolio without changing the amount of money invested in each section. In this case the cost of the derivative to buy can be regarded as a “nonlinear transaction cost”.
The transactions must obey the following natural constraints:

\[
C_j^r k \in [0, 1] \quad \forall r, k \in \{0, \ldots, n\}, \quad \forall j = 0, 1, \ldots,
\]

\[
\sum_{k=0}^{n} C_j^r k \leq 1 \quad \forall r \in \{0, \ldots, n\}, \quad \forall j = 0, 1, \ldots
\]  

(16)

This essentially means that we can move at time \(t_j\) from any section \(r\) no more than its total value \(S_j^{(r)}\). Because of the presence of transaction costs we should also avoid the situation where a section receives and gives money at the same time, as this behavior results in additional costs. A section that receives money should not give money to other sections at the same time and conversely a section that gives money should not receive. This can be expressed by the following conditions:

\[
\begin{align*}
\sum_{k=0}^{n} C_j^r k S_j^{(r)} > 0 & \implies \sum_{k=0}^{n} C_j^r S_j^{(k)} = 0, \\
\sum_{k=0}^{n} C_j^r k S_j^{(k)} > 0 & \implies \sum_{k=0}^{n} C_j^r S_j^{(r)} = 0.
\end{align*}
\]  

(17)

5.2.2. The withdrawals

We turn now to the problem of writing the evolution equations for \(\{S^{(0)}, \ldots, S^{(n)}\}\).

Let’s consider the effect of the withdrawals \(\{p_{t_j}\}\) from the fund: let \(A_{t_j}^+\) the value of the fund right after the withdrawal \(p_{t_j}\), i.e. the right limit \(t \rightarrow t_j^+\), \(t_j \in \{t_1, \ldots, t_p\}\).

Through a generalization of the self-financing portfolio with transaction costs, see [15], we have

\[
A_{t_j}^+ = A_{t_j} - \beta [A_{t_j} - A_{t_{j-1}}^+]^+ \quad t_j \in \{t_1, \ldots, t_p\}.
\]  

(18)

There are a number of choices to have a fund whose value is equal to \(A_{t_j}^+\) right after the withdrawals. We assume that the withdrawals are such that the proportion of each section (i.e. the control) does not change (see Eq. (19)).

5.2.3. The transaction costs

Both transactions and withdrawals entail transaction costs. We assume that:

- the transaction costs associated with the withdrawal from section \(i\) is linearly proportional to the size of the withdrawal through the constant \(\lambda^{(i)}\).
- the transaction cost of moving wealth from section \(S^{(r)}\) to section \(S^{(k)}\) is paid only on buying and not on selling.
- the transaction cost associated with the transaction from section \(r\) to section \(k\) is linearly proportional to the size of the transaction through the constant \(\zeta^{(k)}\).
5.2.4. The evolution equations for the sections

We now turn to writing the evolution equation for the sections \( S^{(i)} \):

- The value \( S^{(i)}_{t_j}^+ \) of the section \( S^{(i)} \) after the withdrawal occurring at time \( t_j \) is, taking into account the transaction costs involved with the withdrawal,

\[
S^{(i)}_{t_j}^+ = S^{(i)}_{t_j} \lambda^{(i)} A_{t_j}^{(i)} - \lambda^{(i)} \left( S^{(i)}_{t_j} - \frac{A_{t_j}^{(i)} S^{(i)}_{t_j}}{A_{t_j}} \right) = S^{(i)}_{t_j} (1 - \alpha^{(i)}_{t_j}),
\]

where we have introduced for sake of simplicity

\[
\alpha^{(i)}_{t_k} = \begin{cases} 
(1 + \lambda^{(i)}) \beta \left( A_{t_k} - A_{t_k-1} \right) & t_k \in \{t_1, \ldots, t_p\}, \\
0 & \text{otherwise},
\end{cases}
\]

which represents the percentage of the investment drawn from the section \( S^{(i)} \) at time \( t_k \).

- After the withdrawal the fund manager can modify the asset-allocation in the fund by modifying the controls \( f^{(c)}_{rk} \).

The total amount invested in each asset after the withdrawal and the portfolio-adjustment evolves according to (11)–(12). Indicating briefly \( S_j = S_{t_j} \) and recalling that \( \xi^{(i)} \) represents the transaction cost for moving wealth from any section to section \( i \), we have:

\[
S^{(i)}_{t_j+1} = f_i \left( S^{(i)}_j - \sum_{k=0}^{n} C^{(k)}_{i} S^{(i)}_j + (1 - \xi^{(i)}) \sum_{k=0}^{n} C^{(k)}_{j} S^{(k)}_j, \Delta W_j \right).
\]

We have for all \( i = 0, \ldots, n \) the evolution equations:

\[
\begin{cases} 
S^{(i)}_{t_j+1} = f_i \left( S^{(i)}_j (1 - \alpha^{(i)}_j) - \sum_{k=0}^{n} C^{(k)}_{j} S^{(i)}_j (1 - \alpha^{(i)}_j) \\
+ (1 - \xi^{(i)}) \sum_{k=0}^{n} C^{(k)}_{j} S^{(k)}_j (1 - \alpha^{(k)}_j), \Delta W_j \right) \\
S^{(i)}_0 \geq 0
\end{cases}
\]

and thus the fund \( A_t \) evolves according to

\[
\begin{cases} 
A_j = \sum_{i=0}^{n} S^{(i)}_j, \\
A_0 = \sum_{i=0}^{n} S^{(i)}_0.
\end{cases}
\]

\(^c\)We are assuming that withdrawals and transactions are made at subsequent times, even though we can allow both to be made at the same time with minor changes in the equations.
Regulation authorities require to limit the percentage of the fund invested in risky assets. The result of this rule is the following state constraint:

\[
\sum_{i=1}^{n} S_j^{(i)} \leq C \cdot \sum_{i=0}^{n} S_j^{(i)}. \tag{23}
\]

In the simple case of one non-risky investment \(U^{(0)}\) and one risky asset \(U^{(1)}\) the equations rewrite:

\[
\begin{align*}
  S_j^{(0)} &= [S_j^{(0)}(1 - \alpha_j^{(0)}) - C_j^{01} S_j^{(0)}(1 - \alpha_j^{(0)})] \\
  &\quad + (1 - \xi_j^{(0)}) C_j^{10} S_j^{(1)}(1 - \alpha_j^{(1)}) \cdot e^{r_j \Delta t_j}, \\
  S_j^{(1)} &= [S_j^{(1)}(1 - \alpha_j^{(1)}) - C_j^{10} S_j^{(1)}(1 - \alpha_j^{(1)})] \\
  &\quad + (1 - \xi_j^{(1)}) C_j^{01} S_j^{(0)}(1 - \alpha_j^{(0)}) \times \exp(\mu_j^{(1)} \Delta t_j + \sigma_j^{(1,0)} \Delta W_j^{(0)} + \sigma_j^{(1,1)} \Delta W_j^{(1)}), \\
  A_j &= S_j^{(0)} + S_j^{(1)}.
\end{align*}
\]

5.3. The reserves and solvency margin

During the lifetime of the fund the company must obey some rules. We assume the following rules based on a real case, but the model does not change if we introduce other constraints:

- The company must check periodically the level of the fund with respect to the minimum guaranteed amount and possibly update the reserves invested in a section \(R\) external to the fund.

  This means that if the fund is under-performing with respect to \(G_t := L_0 e^{r_s t}\) the company must keep the difference \(G_t - A_i\) invested in a risk-free account \(R\). As soon as the value of the fund reaches the minimum guaranteed amount the external section \(R\) can be emptied.

- The company must take part in the investment with its own capital, called “solvency margin”. This is stored in a risk-free section external to the fund \(M\). The solvency margin must be a fixed percentage \(\gamma\) of the liabilities.

It is worth noting that these operations, together with the withdrawals \(\{p_i\}\) usually happen on different time-scale basis. This time lag is profitable for the company as the withdrawals are usually allowed quite often (every three months for example) whereas the restoring in the sections \(R\) and \(M\), that represent a cost for the company as will be clear in a while, are required only once per year.

We assume that the withdrawals happen at every \(t_j \in \{t_1, \ldots, t_p\}\), the restoring in the section \(R\) are updated at every \(s_j \in \{s_1, \ldots, s_r\}\) and the solvency margin \(M\) is updated for \(z_j \in \{z_1, \ldots, z_m\}\) and \(t_p = s_r = z_m = T\) where some of these times
may coincide. We have
\[
\begin{aligned}
R_{s_j} &= [A_{s_j} - G_{s_j}]^{-} \quad s_j \in \{s_1, \ldots, s_r\}, \\
R_0 &= 0,
\end{aligned}
\]  
(25)
with \([x]^- = -\min(x, 0)\) the negative part of \(x\), and
\[
\begin{aligned}
M_{z_j} &= \gamma A_{z_j} \quad z_j \in \{z_1, \ldots, z_m\}, \\
M_0 &= \gamma A_0 = \gamma L_0.
\end{aligned}
\]  
(26)

We want to quantify the cost of the external sections \(R\) and \(M\). Consider for example the section \(R\): we assume that the capital to be invested has to be borrowed from a virtual section \(B(R)\) at rate \(r_1(t)\), whereas the investment \(R\) provides a rate \(r_2(t)\). Of course if \(r_1 = r_2\) the cost of borrowing money for the company would be zero.

To minimize the cost of borrowing money from the section \(B^{(R)}\) the company will try to return as soon as possible the borrowed amount. We have thus a balance equation between the "lending compartment" \(R\) and the "borrowing compartment" \(B(R)\) in which every residual debt grows up at rate \(r_1\). The value \(B^{(R)}\) of \(B(R)\) at time \(s_j \in \{s_1, \ldots, s_r\}\) consists of the residual debt at time \(s_{j-1}\) plus the borrowed amount at time \(s_j\). Assuming for sake of simplicity \(r_1\) and \(r_2\) constant we have:
\[
\begin{aligned}
B_j^{(R)} &= B_{j-1}^{(R)} e^{r_1 \Delta s_{j-1}} + (R_j - R_{j-1} e^{r_2 \Delta s_{j-1}}), \\
B_0^{(R)} &= 0.
\end{aligned}
\]  
(27)
The cost at time \(T\) of this external section is equal to the final value of the section \(B^{(R)}\):
\[
V_T^{(R)} = B_{s_r}^{(R)} = \sum_{j=1}^{T} (R_j - R_{j-1} e^{r_2 \Delta s_{j-1}}) e^{r_1 (T-s_j)},
\]  
(28)
not to be confused with \(C_{j}^{i\rightarrow k}\) that represents the transaction occurring at time \(t_j\) from section \(i\) to section \(k\). It is easy to check that
\[
V_T^{(R)} = R_T + C(r_1, r_2, \{R_j\}_{j=0,\ldots,r-1}),
\]
where \(R_T\) is the final restoring at time \(T\) and the last term is the accrued debt over \([0, T]\). It is straightforward proving that this debt is always non-negative as \(r_2 \leq r_1\) and
\[
r_1 = r_2 \implies C(r_1, r_2, \{R_j\}_{j=0,\ldots,r-1}) = 0 \quad \text{for any } \{R_j\}_{j=0,\ldots,r-1}
\]  
(29)
in agreement with its financial meaning.

In analogy with the section \(R\) the section \(M\) will provide a rate \(\tilde{r}_2(t) \leq \tilde{r}_1(t)\), with \(\tilde{r}_1\) the borrowing rate from a virtual section \(B^{(M)}\). The only difference with
the previous case being the fact that the final amount $M_T$ of the section can be recovered by the company. Assuming again $\bar{r}_i$ constant the final cost becomes:

$$V_T^{(M)} = M_{zm} - B_{zm}^{(M)} = \gamma A_{zm} - \gamma A_0 e^{\bar{r}_i T} - \sum_{j=1}^{m} (M_j - M_{j-1} e^{\bar{r}_2 \Delta z_j - 1}) e^{\bar{r}_i (T - z_i)}, \quad (30)$$

where again $V_T^{(M)} \geq 0$ as $\bar{r}_2 \leq \bar{r}_1$ and

$$\bar{r}_1 = \bar{r}_2 \implies V_T^{(M)} = 0 \quad \text{for any } \{M_j\}_{j=0,...,m}. \quad (31)$$

### 5.4. The performance index

The main goals for the company are represented by the shortfall risk-minimization and the maximization of the profit for the shareholders. However it is important to consider also the costs coming, in a long-term setting, from a swinging yield or from being too much time below the minimum guaranteed return.

The objectives of an efficient management can be summarized in the following points.

- Protection against the shortfall risk of the final payoff: the company should manage its position in order to maximize the probability of $A_T$ being greater than $G_T$. In the real management anyway this constraint is not too strong: the company would be glad to accept a little loss at maturity if the profit coming from the withdrawals is quite high. For this reason a strong constraint of the type “$A_T \geq G_T$ almost surely” could be misleading, and it could be more reasonable to introduce one more term in the functional to maximize.

- Maximization of the profit for shareholders, i.e. maximization of the profit coming from the periodical withdrawals from the fund.

- Control on the fund fluctuations: the company should try to keep the annual yield of the fund constant and positive as a constant low yield is often more appealing than a highly oscillatory yield because of the possible risk-aversion of shareholders.

- Control on the annual position of the fund with respect to the minimum guaranteed amount, as an under-performing fund could result in a potential loss of the client base, that could move towards other guaranteed-investment products that could provide higher returns.

Some of these goals may become more important than others in certain periods. We should therefore provide a parametric and highly tunable model to let the company privilege from time to time one or more of these objectives with respect to others. To achieve this goal it is convenient to introduce a fairly general cost functional which depends on the various factors that influence the management and its valuation.

We assume that the factors to take into account in the dynamic optimization procedure are the following:
(1) The net profit at time $T$ for the company: $U_T = P_T + V_T^{(M)} - V_T^{(R)}$ (recall that $P_T$ is the company pay-out given by (7)).

(2) The average yield $\frac{A_T - A_0}{A_0}$.

(3) The annual yield $\frac{A_j - A_{j-1}}{A_{j-1}}$ for every $j = 1, \ldots, T$.

(4) The position with respect to the minimum guaranteed pay-out: $\frac{A_j - G_j}{G_j}$, $j = 1, \ldots, T$.

Since these factors are of different nature we need to measure each of them with a separate utility function (that will be called $g_0, \ldots, g_3$ respectively) chosen according to the objectives of the management. The following functional is then to be maximized over all the admissible controls, i.e. over all the sequences $\{C_{rk}\}$ satisfying (16), (17) and (23) (we will refer to this set as $C$).

$$J(A_0, \{C_{rk}\}) := \mathbb{E} \left[ g_0(U_T) + g_1 \left( \frac{A_T - A_0}{A_0} - \eta_1 \right) + \sum_{j=1}^{T} g_2 \left( \frac{A_j - A_{j-1}}{A_{j-1}} - \eta_2 \right) + \sum_{j=1}^{T} g_3 \left( \frac{A_j - G_j}{G_j} - \eta_3 \right) \right]. \quad (32)$$

The parameters $\eta_1, \ldots, \eta_3$ together with the utility functions $g_0, \ldots, g_3$ must be chosen to achieve a given objective: by balancing such parameters the management can choose whether to privilege the maximization of the profit, the minimization of fund oscillations, the control on the position with respect to the guarantee and so on.

The optimal investment strategy (if such a strategy exists) is the sequence $\{C_{rk}\}_j$ of portfolio adjustments such that:

$$v(A_0) = \max_{\{C_{rk}\} \in C} J(A_0, \{C_{rk}\}) = J(A_0, \{C_{rk}\}). \quad (33)$$

### 5.5. Additional constraints

It is also possible to exclude from the optimization the trajectories that do not satisfy some probabilistic constraints, to allow for a deeper control on the admissible trajectories:

$$\begin{cases} 
P \left( \frac{A_T - A_0}{A_0} \leq \bar{\eta}_1 \right) < \bar{\varepsilon}_1, \\
P \left( \frac{A_j - A_{j-1}}{A_{j-1}} \leq \bar{\eta}_2 \right) < \bar{\varepsilon}_2 \quad \forall j, \\
P \left( \frac{A_j - G_j}{G_j} \leq \bar{\eta}_3 \right) < \bar{\varepsilon}_3 \quad \forall j, \\
P \left( \frac{A_{T\infty} - A_0}{A_0} \leq \bar{\eta}_{\infty} \right) = 0,
\end{cases} \quad (34)$$
where the last one requires that we can start again our optimization from $T$ and reach the further maturity $T_{\infty}$ with a yield not less than $\bar{\eta}_{\infty}$.

The correct combination of probabilistic constraints and correction terms in the cost functional makes it possible to select the best strategy in order to achieve a given objective.

6. Numerical Tests

In this section we present some preliminary results produced by a software prototype which implements the model described above. For each scenario the prototype determines the controls that maximize the utility function taking into account the solvency condition and other possible constraints. For the optimization we started using a deterministic *Newton-like* algorithm (Truncated Newton) [16]. This is a "local" algorithm which requires information on the structure and the regularity of the function to be maximized. The main advantages are the low computational cost and the high order of convergence. The time required to find the optimal controls for a typical scenario is less than 1.5 seconds on a 800 MHz PC running LINUX.

![Fig. 3. The results of the optimization using a Newton-like (i.e., deterministic) algorithm and the simulated annealing. A positive transaction means that bonds are sold to buy stock. A negative transaction means that stock is sold to buy bonds. The transactions are normalized with respect to the total amount (e.g., 0.5 means half of the bonds are sold).](image-url)
Unfortunately, we have found that local algorithms may fail to find the optimum as can be easily seen looking at the Fig. 3. In the upper part we show a possible scenario for the returns of a risk-free (bond) and a risky (stock) asset. The scenario has been generated simple enough so that the optimum can be determined “by hand”. The resulting optimal transactions between the two sections of the fund are reported in the middle plot of Fig. 3. In the lower part of the panel we show the results produced by the Newton-like (“TN”) optimization procedure. It is apparent that the Newton algorithm does not provide a real “optimal” result probably because it finds a local minimum.

If a deterministic algorithm fails in a simple case like this, it is necessary to resort to probabilistic algorithms. The most widely known of such algorithms is the Simulated Annealing (SA) [17]. The SA has, in general, a high computational cost but it is able, with a careful tuning of the annealing procedure, to escape from local minima.

This is confirmed by the results shown in the same plot of Fig. 3. The optimization procedure provides the right answer using the SA but, as expected, the

![Graph showing the unitary value of the stock and risk free investments in the scenario used for the comparison between the optimization and the hedging approach. The unitary value of the minimum guaranteed return investment is also shown.](image)
computational cost grows significantly. The time required by the SA to find the optimum for a single scenario is, on average, about 30 seconds on the same 800 MHz PC where the deterministic algorithm takes 1.5 seconds.

We present now a comparison between the results produced by our probabilistic optimization approach, a hedging technique and a lazy policy in which there are no transactions among the fund sections (that is, the fund composition is decided once and for all at the beginning). For the sake of simplicity we consider a single scenario shown in Fig. 4. The choice is restricted to either a risky asset (stock) or a risk-less investment (bond). The hedging approach consists of investing in the risk-less asset an amount of money equal to the $\Delta$ of the virtual option owned by the policy holder.

The optimization procedure aims to maximize the final value $A_T$ of the fund taking into account transaction costs and solvency constraints.

The performance of the fund for the three different management policies is shown in Fig. 5. Although the hedging technique fulfills the solvency requirement (the total return of the fund is higher than the minimum guaranteed return) it is clear that the fund managed according to the optimization approach performs much better. The problem is that the hedging technique does nothing until the value of the fund falls below the minimum guaranteed.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5}
\caption{The unitary value of the fund for three different management policies. The minimum guaranteed return is shown to ease the comparison.}
\end{figure}
The aim of the next set of tests is to highlight some features of the model. For the sake of simplicity we limit ourselves to the bi-dimensional case, \( n = 1 \), where \( S^{(0)} = B \) is the risk-less investment and \( S^{(1)} = S \) is the risky investment. The utility function \( g_0 \) used in these experiments is the following Hyperbolic Absolute Risk Aversion (HARA) function

\[
g_0(x) = \frac{x^{1-q}}{1-q}
\]

with an absolute risk-aversion coefficient \( q = 0.7 \). This utility function expresses, the risk-aversion of the shareholders \((g''_0(x) = -qx^{-q-1} < 0)\), their decreasing

<table>
<thead>
<tr>
<th>Table 1. Parameters used in the simulations.</th>
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<tbody>
<tr>
<td>( C = 0.5 )</td>
</tr>
<tr>
<td>( S_0 = 5 )</td>
</tr>
<tr>
<td>( \lambda^S = 0.01 )</td>
</tr>
<tr>
<td>( r_1 = 0.06 )</td>
</tr>
<tr>
<td>( \mu_S = 0.16 )</td>
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</table>

Fig. 6. Scenario for the evolution of the bond and stock prices.
absolute risk-aversion \( A'(x) = -qx^{-2} < 0 \) and their constant relative risk aversion \( R'(x) = 0 \).

To ease the presentation we do not include in these simulations the correction terms defined in (32). In this case the functional that we consider becomes:

\[
J(A_0, \{C^r_j\}) = g_0(U_T)
\]

over the set \( C \) of the admissible control policies. We recall that the amount \( U_T \) represents the net profit at time \( T = N\Delta t \) (see Sec. 5.4).

We have fixed in the following tests \( \Delta t_j = \Delta z_j = \Delta s_j = 0.25, \, N = 20 \), that corresponds to a global adjustment of all the sections \( (S, B, R, M) \) every three months, over a period of \( T = N\Delta t \) equal to 5 years, with the withdrawals from the fund (2) performed every three months. The other parameters have been chosen as in Table 1. In Fig. 6 is shown the evolution of the prices \( U^B_j \) and \( U^S_j \). This scenario is particularly interesting since the corresponding bond and stock yields intersect each other more than once. In this way it is easy to test how the optimizer “reacts” to changes in the market situation.

Fig. 7. The optimal percentage of stocks in portfolio as a function of the transaction costs.
To represent the control policy, we use the percentage of stocks (or bonds) in portfolio at each time step. In the tests, the values of the external sections $R$ and $M$, the profit $P$, the functional $J$ and all the costs are expressed as absolute numbers (“money”). To interpret the data consider that the initial allocation is $A_0 = 45$, then, for example, a cost $V_T^{(R)} = 4.5$ means 10% of the initial investment.

The transaction costs play a crucial role in a realistic model. They act as friction terms increasing the inertia towards allocation changes. When the transaction costs exceed a given amount, that depends on the scenario and on all the other parameters, some transactions are not performed anymore. The more the transaction costs increase, the less allocation changes take place. The controls are then grouped into clusters. In Fig. 7 it can be seen (left) that the optimal control for $\lambda^S = \lambda^B = 0$ is such that the allocation changes at time 0.75, 1.50, 2.75, 3.25 and 4.25. If $\lambda^S = \lambda^B = 0.04$ the only allocation change occurs at time 3.25 (actually few more transactions are needed, but only to fulfill the state constraint). For transaction costs higher than 0.04 no allocation change is performed and an average allocation is kept for the whole optimization period.

Figure 8 shows the value of the fund (left) and the net profit (right) as a function of $\lambda^S = \lambda^B$.

In the last example we wanted to evaluate the effect of the state constraint on the net profit of the company. To this purpose we have compared the case without state constraint to a case in which $C = 0.5$, that is, only half of the total wealth can be invested in risky assets at any time $t \in [0, T]$ (all the other parameters are fixed as in Table 1).

As a consequence of the limited freedom of the company, the value of the profit when $C = 0.5$ is lower with respect to the case with no state constraint, as depicted in Fig. 9.

The lower profit in this case is mainly due to the higher cost of managing the reserves, that is not compensated by the lower margin cost.
7. Conclusions and Further Developments

In this paper we have formulated and described a model for the optimal asset-liability management of insurance companies. This model was built on demand of the I.N.A. insurance company that aimed to support the management in choosing the “best” asset-allocation policy, satisfying regulatory and solvency constraints and leading on to the achievement of a given objective.

We focused on a typical guaranteed-investment contract taking into account the actual management of the fund, the discrete structure of the withdrawals, the effect of the time-scale gap between withdrawals and restoring, regulatory constraints, solvency constraints and transaction costs. The result is a parametric model in which the “cost functional” takes into account the aims of the management whereas the constraints (both of deterministic and probabilistic nature) limit the maximization over the set of the “admissible” strategies.

The first numerical tests confirm that the optimization problem is by far from being trivial so probabilistic algorithms have been employed to avoid local minima.

This model is the departure point of various directions of research we are dealing with.
An implementation of a “dynamic” discrete-time stochastic optimization model. Although the “static” approach provides a quick tool to test the reliability of the model, it can lead on to a large number of variables if the number of time-steps is high. The “dynamic” approach is a bit more delicate but does not suffer this limitation.

A continuous-time formulation and treatment of the above model which employs all the power of stochastic calculus and stochastic dynamic-programming. This leads to non-linear second order degenerate-parabolic differential equations that usually do not admit an explicit solution and an efficient numerical algorithm is thus needed.

References
