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THE VALUATION OF BONDS AND BOND OPTIONS: SOME EMPIRICAL TESTS

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SUMMARY

This paper presents a general method for valuing fixed rate bonds and options written on them. In the first part, of a theoretical nature, we present valuation formulae, derived within the framework of the Cox, Ingersoll and Ross (CIR) model, both for bonds and for European options written on bonds (with and without coupons) and yields. We recall the theoretical parity of put and call options. In the second part we describe a procedure that can be used to estimate the CIR model using the prices of riskless coupon bonds. Lastly, we describe the valuation of bonds denominated in three different currencies and of the options implied in some Italian government securities.

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1. INTRODUCTION

The valuation of the different contracts referring to bonds -- spot contracts, forward or futures contracts, repos, the different types of options on prices and yields (call, put, etc.), swaps, caps and floors, etc. -- cannot be based on a series of *ad hoc* methods. A general approach nonetheless requires a theoretical model that formalizes, with an appropriately small number of parameters, the influence of the most important factors and the random components.

The model used in this paper is the one-factor version of that proposed by Cox, Ingersoll and Ross (CIR). According to this model, there exists a "pivot rate" that determines the whole term structure of interest rates: the instantaneous rate, i.e. the rate on an investment of instantaneous duration. The dynamics of the instantaneous rate are of the mean-reverting type: even if disturbed by shocks, the rate tends to return towards a long-run average level. The market equilibrium condition is embodied in the local expectations hypothesis: the expected return in the next instant of time on an investment in bonds of any maturity is equal to the instantaneous rate plus a risk premium/discount (the amount of risk is equal to the elasticity of the price with respect to the instantaneous rate).

The model describing the dynamics of bond prices is more complex than that generally adopted for share prices (geometric Brownian motion). This is because the prices of bonds, unlike those of shares, depend on their maturity. It follows that the valuation of bond options is more complex than that of share options. In order to value bond options, it is necessary to estimate four parameters instead of just one (volatility in the case of the Black and Scholes formula).

In this paper we describe a general method for valuing fixed interest bonds and options written on them. In the first part, of a theoretical nature, we present the valuation formulae derived within the framework of the CIR model for both bonds and European options written on bonds (with and without coupons) and yields (§2). We also show the theoretical parity of call and put options. In the second part we describe a procedure that can be used to estimate the CIR model using the prices of riskless coupon bonds (§3). In the following section we examine the valuation of bonds denominated in three different currencies and of the options implied in some Italian government securities (§4).

2. THEORETICAL ASPECTS

2.1 Description of the CIR model

The one-factor version of the Cox, Ingersoll and Ross (CIR) model assumes that:

1. the instantaneous interest rate (r) follows the process described by the stochastic differential equation

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dz \quad (1)$$

where κ is the speed of adjustment of the interest rate r towards its long-run average θ , $\sigma\sqrt{r}$ is the volatility of the changes in r and dz is a standardized Wiener process; if $0 < \kappa < 1$ the instantaneous rate converges towards its average value θ (a mean-reverting process);

2. the expected instantaneous return on bonds of any maturity is equal to the interest rate r plus a risk premium (the local expectations hypothesis):

$$E\left(\frac{dP}{P}\right) = \left(r + \lambda r \frac{P_r}{P}\right)dt \quad (2)$$

where $P = P(r, t, T)$ is the price of a pure discount bond with residual maturity $\tau = T - t$, $-\lambda$ is the market price of risk and rP_r/P is the elasticity of the bond's price with respect to r ;

3. the usual perfect market assumptions apply.

Under these assumptions it can be shown that if the value $H = H(r, t, T)$ of a financial asset depends exclusively on the current term structure of interest rates, it will follow (in the interval between two successive coupons) the stochastic differential process defined by (3):

$$\frac{1}{2}\sigma^2 r H_{rr} + [\kappa(\theta - r) - \lambda r] H_r + H_t - rH = 0. \quad (3)$$

This equation can be solved once appropriate boundary conditions are imposed on $H(r, t, T)$, according to the specific features of each type of financial asset.

2.2 Pure discount bonds

For the price $P(r, t, T)$ of a pure discount bond with residual maturity $\tau = T - t$, the solution of equation (3), subject to the boundary condition

$$P(r, T, T) = 1 \quad (4)$$

is

$$P(r, t, T) = F(t, T) e^{-G(t, T)r} \quad (5)$$

where

$$F(t, T) = \left[\frac{\phi_1 e^{\phi_2 \tau}}{\phi_2 (e^{\phi_1 \tau} - 1) + \phi_1} \right]^{\phi_3}$$

and

$$G(t, T) = \frac{e^{\phi_1 \tau} - 1}{\phi_2 (e^{\phi_1 \tau} - 1) + \phi_1}.$$

Equation (5) gives the value of a pure discount bond with residual maturity τ as a function of the state variable r and of the three parameters ϕ_1 , ϕ_2 and ϕ_3 , where:

$$\phi_1 = \sqrt{(\kappa + \lambda)^2 + 2\sigma^2} \quad (6)$$

$$\phi_2 = \frac{\kappa + \lambda + \phi_1}{2} \quad (7)$$

$$\phi_3 = \frac{2\kappa\mu}{\sigma^2}. \quad (8)$$

Equation (5) can be used to derive the whole term structure of interest rates $R(r, t, T)$ by using the following relationship:

$$R(r, t, T) = \frac{-\ln[P(r, t, T)]}{T - t}. \quad (9)$$

In particular:

$$R_0 = \lim_{T \rightarrow t} R(r, t, T) = r \quad (10)$$

$$R_\infty = \lim_{T \rightarrow \infty} R(r, t, T) = (\phi_1 - \phi_2)\phi_3. \quad (11)$$

The parameter σ of equation (1) can be determined using the relationship:

$$\sigma = \sqrt{2(\phi_1\phi_2 - \phi_2^2)}. \quad (12)$$

The economic significance of the three parameters ϕ_1 , ϕ_2 and ϕ_3 is clear when the value of R_∞ is substituted for θ . In this case we have:

$$\phi_1 = \kappa - \lambda \quad (13)$$

$$\phi_2 = \kappa \quad (14)$$

$$\phi_3 = -\frac{\theta}{\lambda}. \quad (15)$$

The derivatives of P with respect to r , ϕ_1 , ϕ_2 and ϕ_3 are as follows:

$$\frac{\partial P}{\partial r} = -PG \quad (16)$$

$$\begin{aligned} \frac{\partial P}{\partial \phi_1} = & \phi_3 \left[\frac{\phi_1 e^{\phi_2 \tau}}{\phi_2 (e^{\phi_1 \tau} - 1) + \phi_1} \right]^{\phi_3 - 1} \left\{ \frac{e^{\phi_2 \tau} [\phi_2 (e^{\phi_1 \tau} - 1) + \phi_1] - \phi_1 e^{\phi_2 \tau} (\phi_2 e^{\phi_1 \tau} \tau + 1)}{[\phi_2 (e^{\phi_1 \tau} - 1) + \phi_1]^2} \right\} e^{-Gr} + \\ & - Pr \frac{e^{\phi_1 \tau} \tau [\phi_2 (e^{\phi_1 \tau} - 1) + \phi_1] - (e^{\phi_1 \tau} - 1) (\phi_2 e^{\phi_1 \tau} \tau + 1)}{[\phi_2 (e^{\phi_1 \tau} - 1) + \phi_1]^2} \end{aligned} \quad (17)$$

$$\begin{aligned} \frac{\partial P}{\partial \phi_2} = & \phi_3 \left[\frac{\phi_1 e^{\phi_2 \tau}}{\phi_2 (e^{\phi_1 \tau} - 1) + \phi_1} \right]^{\phi_3 - 1} \left\{ \frac{\phi_1 e^{\phi_2 \tau} \tau [\phi_2 (e^{\phi_1 \tau} - 1) + \phi_1] - \phi_1 e^{\phi_2 \tau} (e^{\phi_1 \tau} - 1)}{[\phi_2 (e^{\phi_1 \tau} - 1) + \phi_1]^2} \right\} e^{-Gr} + \\ & + Pr \left[\frac{e^{\phi_1 \tau} - 1}{\phi_2 (e^{\phi_1 \tau} - 1) + \phi_1} \right]^2 \end{aligned} \quad (18)$$

$$\frac{\partial P}{\partial \phi_3} = F \ln \left[\frac{\phi_1 e^{\phi_2 \tau}}{\phi_2 (e^{\phi_1 \tau} - 1) + \phi_1} \right] e^{-Gr}. \quad (19)$$

2.3 Coupon bonds and the stochastic duration

The price of a coupon bond is obtained simply as the sum of the bond's payments, with each payment multiplied by the price of the corresponding pure discount bond:

$$B(r, t; \mathbf{s}, \mathbf{a}) = \sum_{j=1}^m a_j e^{-R(r, t, s_j)(s_j - t)} = \sum_{j=1}^m a_j P(r, t, s_j) \quad (20)$$

where:

$B(r, t, \mathbf{s}, \mathbf{a})$ is the price at time t of a coupon bond;

$\mathbf{a}$ is the vector of payments a_j ($j = 1, 2, \dots, m$);

$\mathbf{s}$ is the vector of payment dates s_j ($t < s_j \leq s_m$);

$R(r, t, s_j)$ is the interest rate for maturity s_j ;

$P(r, t, s_j)$ is the price of a zero-coupon bond with maturity s_j .

In the CIR model the basis risk associated with a pure discount bond, i.e. the proportional change in the price determined by an infinitesimal change in the instantaneous interest rate (with the sign changed), is given by equation (5) as:

$$-\frac{P_r}{P} = G(\tau) \quad (21)$$

The basis risk of a zero-coupon bond is thus equal to the weighted average of the basis risks of the pure discount bonds with maturities coinciding with the various payment dates, using the discounted values of the payments as weights:

$$\frac{-B_r}{B} = \frac{-\sum aP_r}{\sum aP} = \frac{\sum aPG(\tau)}{\sum aP} \quad (22)$$

This measure of the risk associated with coupon bonds can be expressed in units of time. In this case Cox, Ingersoll and Ross (1979) used the term “stochastic duration”. The stochastic duration (τ) of a coupon bond is equal to the maturity of the pure discount bond with the same degree of risk:

$$\tau = G^{-1}(B_r / B) = \frac{1}{\phi_1} \ln \left(1 - \frac{B_r / B \phi_1}{1 + B_r / B \phi_2} \right) \quad (23)$$

2.4 Options on pure discount bonds

Let $c(r, t, T; s, K)$ be the value at time t of a call option, with striking price K and maturity T , written on a pure discount bond with maturity s ($s > T > t$). The price of the option, obtained by solving equation (3), subject to the boundary condition

$$c(r, T, T; s, K) = \max[P(r, T, s) - K, 0] \quad (24)$$

is

$$C(r, t, T; s, K) = P(r, t, s) \chi^2(d_1, df_1, nc_1) - KP(r, t, T) \chi^2(d_2, df_2, nc_2) \quad (25)$$

where $\chi^2(d, df, nc)$ is the non-central chi-square distribution function,¹ with df degrees of freedom and non-centrality parameter nc , valued at point d , and, respectively, according to Cox, Ingersoll and Ross (1985) and Jamshidian (1990):

$$d_1 = 2r^*[\varphi + \psi + G(T, s)] \quad d_1 = r^*[v + 2G(T, s)] \quad (26)$$

$$d_2 = 2r^*(\varphi + \psi) \quad d_2 = r^*v \quad (27)$$

$$df_1 = df_2 = 2\phi_3 \quad df_1 = df_2 = 2\phi_3 \quad (28)$$

$$nc_1 = \frac{2\varphi^2 r e^{\phi_1(T-t)}}{\varphi + \psi + G(T, s)} \quad nc_1 = \frac{vbr}{v + 2G(T, s)} \quad (29)$$

$$nc_2 = \frac{2\varphi^2 r e^{\phi_1(T-t)}}{\varphi + \psi} \quad nc_2 = br \quad (30)$$

$$\varphi = \frac{2\phi_1}{\sigma^2 [e^{\phi_1(T-t)} - 1]} \quad v = \frac{4}{\sigma^2 G(t, T)} \quad (31)$$

$$\psi = \frac{2\phi_2}{\sigma^2} \quad b = vG'(t, T) \quad (32)$$

$$r^* = \frac{\ln \left[\frac{F(T, s)}{K} \right]}{G(T, s)} \quad r^* = \frac{\ln \left[\frac{F(T, s)}{K} \right]}{G(T, s)} \quad (33)$$

¹. For the calculation of the non-central chi-square distribution function, see Farebrother (1987) and Ashour and Abdel-Samad (1990).

The value r defined by equation (33) is the critical interest rate below which the call option will be exercised and is obtained by putting $P(r^*, T, s) = K$ and solving with respect to r^* .

A call option, with striking price K and maturity T , written on a pure discount bond with maturity s ($s > T > t$) can thus be replicated by a portfolio constructed by buying $\chi^2(d_1, df_1, nc_1)$ pure discount bonds with maturity s and selling $K\chi^2(d_2, df_2, nc_2)$ pure discount bonds with maturity T .

Put-call parity.

Let $p(r, t, T; s, K)$ be the value at time t of a put option, with striking price K and maturity T , written on a pure discount bond with maturity s . The value of the put option can be obtained by way of the put-call parity, which, in the case of options on pure discount bonds, is given by:

$$c(r, t, T; s, K) = p(r, t, T; s, K) + P(r, t, s) - KP(r, t, T). \quad (34)$$

Such options are markedly different from those written on shares. In particular, the values of a call and a put can move in the same direction when the price of the underlying bond changes and they can also be inversely correlated with the maturity.

2.5 Options on coupon bonds

With the CIR model, as with all one-factor models, the option on a coupon bond is equivalent to a portfolio of options on pure discount bonds [Jamshidian (1989)]²

$$\begin{aligned} c(r, t, T; \mathbf{s}, \mathbf{a}, K) &= \sum_{j=q}^m a_j c(r, t, T; s_j, K_j, 0) = \\ &= \sum_{j=q}^m a_j [P(r, t, s_j) \chi^2(d_{1,j}, df_{1,j}, nc_{1,j})] - KP(r, t, T) \chi^2(d_2, df_2, nc_2) \end{aligned} \quad (35)$$

where a_j is the bond's payment at time s_j ($T < s_q \leq s_j \leq s_m$), r^* is the solution of $B(r^*, T; \mathbf{s}, \mathbf{a}) = K$ and $K_j = P(r^*, T, s_j)$.

A call option, with striking price K and maturity T , written on a coupon bond with maturity s ($s > T > t$) can thus be replicated by a portfolio constructed by buying $a_j \chi^2(d_{1,j}, df_{1,j}, nc_{1,j})$ pure discount bonds with maturity s_j ($j = m, \dots, n$) and selling $K_j \chi^2(d_{2,j}, df_{2,j}, nc_{2,j})$ pure discount bonds with maturity T .

Put-call parity.

The value at time t of a put option, with striking price K and maturity T , written on a coupon bond with payments $\mathbf{a}$ at dates $\mathbf{s}$, can be obtained by way of the put-call parity, which, in the case of options on coupon bonds, is given by:

$$c(r, t, T; \mathbf{s}, \mathbf{a}, K) = p(r, t, T; \mathbf{s}, \mathbf{a}, K) + B(r, t; \mathbf{s}, \mathbf{a}) - B(r, t; \mathbf{s}, \mathbf{a}') - KP(r, t, T) \quad (36)$$

where:

$$\mathbf{a}' = \begin{cases} \mathbf{a} & \text{se } s_j < T \\ 0 & \text{se } s_j \geq T. \end{cases}$$

2.6 Options on yields

Let $c_i(r, t, T; s, K_i)$ be the value at time t of a call, with striking rate K_i and maturity T , written on the yield $i(r, T, s) = 1/P(r, T, s) - 1$ relative to the period (T, s) ($s > T > t$). The price of the option is obtained by solving equation (3), subject to the boundary condition

². Longstaff (1990a) obtained the same formula using a different approach..

$$c_i(r, T, T; s, K_i) = \max\{[1/P(r, T, s) - 1] - K_i, 0\} \quad (37)$$

Equation (3) can be solved using numerical methods. It can be rewritten as a function of the parameters ϕ_1 , ϕ_2 and ϕ_3 , as well as of the state variable r , by using the definitions (6), (7) and (8). Referring to equation (12), we have:

$$\frac{1}{2}\sigma^2 r H_{rr} + \left[\frac{1}{2}\sigma^2 \phi_3 - (2\phi_2 - \phi_1)r \right] H_r + H_t - rH = 0 \quad (38)$$

The solution of the differential equation (38), subject to the boundary condition (37), can be approximated using numerical algorithms.

Put-call parity.

The value at time t of a put option, with striking rate K_i and maturity T , written on the yield relative to the period (T, s) ($s > T > t$) can be obtained by way of the put-call parity, which, in the case of options on yields, is given by:

$$c_i(r, t, T; s, K_i) = p_i(r, t, T; s, K_i) + c_i(r, t, T; s, 0) - K_i P(r, t, T) \quad (39)$$

A special case

If the settlement of the final value of options on yields is not made at maturity, but deferred for a period equal to that to which the optionable interest rate refers, the price of the options can be obtained explicitly using (25) and (34).

In this case the final value at time T of a call option, with striking rate K_i , maturity T and settlement at time s , written on the yield relative to the period (T, s) ($s > T > t$) is:

$$\begin{aligned} c_i(r, T, T; s, K_i) &= \max\{[1/P(r, T, s) - 1] - K_i, 0\} P(r, T, s) = \\ &= (1/K) \max\{K - P(r, T, s), 0\} \end{aligned} \quad (40)$$

where $K = 1/(1 + K_i)$. Accordingly, we have:

$$c_i(r, t, T; s, K_i) = (1 + K_i) p\left(r, t, T; s, \frac{1}{1 + K_i}\right) \quad (41)$$

Analogously:

$$p_i(r, t, T; s, K_i) = (1 + K_i) c\left(r, t, T; s, \frac{1}{1 + K_i}\right) \quad (42)$$

The put-call parity for such options is:

$$c_i(r, t, T; s, K_i) = p_i(r, t, T; s, K_i) - (1 + K_i) P(r, t, s) + P(r, t, T). \quad (43)$$

In addition, the value at time t of a forward contract, with forward rate K_i , maturity T and settlement at s , written on the yield relative to the period (T, s) ($s > T > t$) is:

$$\begin{aligned} F(r, t, T; s, K_i) &= c_i(r, t, T; s, K_i) - p_i(r, t, T; s, K_i) = \\ &= P(r, t, T) - (1 + K_i) P(r, t, s). \end{aligned} \quad (44)$$

The value at time t of this contract is null if the forward rate K_i is equal to that implied in the term structure of interest rates,

$$K_i = \frac{P(r, t, T)}{P(r, t, s)} - 1 \quad (45)$$

Options on continuously compounded interest rates

In the case of options written on the yields $R(r, t, T)$ defined by equation (9), Longstaff (1990b) obtained the following closed-form formula:³

$$c_R(r, t, T; s, K_i) = P(r, t, T) [R_\tau Q(d, df + 4, nc) \xi - K_R Q(d, df, nc) + \psi / \tau] \quad (46)$$

where $Q(d, df, nc) = 1 - \chi^2(d, df, nc)$ is the non-central chi-square complementary distribution function and:

$$d = \frac{4\{K_R \tau + \ln[F(\tau)]\}}{\sigma^2 G(\tau) G(T)} \quad (47)$$

$$df = 2\phi_3 \quad (48)$$

$$nc = \frac{4\phi_1^2 e^{\phi_1 T} G(T) \{\ln[F(\tau)] + R_\tau \tau\}}{\sigma^2 \left(e^{\phi_1 T} - 1\right)^2 G(\tau)} \quad (49)$$

$$\xi = \left[\phi_1 G(T) \left(e^{\phi_1 T} - 1\right)^{-1} e^{\phi_1 T/2} \right]^2 \quad (50)$$

$$\begin{aligned} \psi = & -\ln[F(\tau)] Q(d, df, nc) + \phi_3 \frac{\sigma^2}{2} G(T) G(\tau) Q(d, df + 2, nc) + \\ & + \xi \ln[F(\tau)] Q(d, df + 4, nc). \end{aligned} \quad (51)$$

The analytical properties of these options are very different from those usually encountered. In particular, call options can have a value that is higher than the underlying interest rate or less than their intrinsic value. The use of the Black and Scholes formula to value options written on interest rates, and hence also caps and floors, can therefore lead to erroneous results, since it is sometimes impossible to generate the correct theoretical value, no matter how large is the adjustment of the variance parameter.

3. ESTIMATION METHODS

3.1 The one-stage approach

The Cox, Ingersoll and Ross model has been estimated using the one-stage approach suggested by Brown and Dybvig (1986). It is assumed, for each of the n coupon bonds observed at a certain date, that:

$$B_i^e = B_i(r, t, \mathbf{s}, \mathbf{a}) + u_i \quad i = 1, 2, \dots, n. \quad (52)$$

where B_i is the actual (cum-coupon) price of the i -th security B_i theoretical price, $\mathbf{s}$ the vector of the m payment dates $\mathbf{a}$ the vector of the payments and u_i , the error term.

Using equation (20), equation (52) can be written as follows:

$$B_i^e = \sum_{j=1}^m a_j P(t, s_j, \boldsymbol{\beta}) + u_i \quad i = 1, 2, \dots, n. \quad (53)$$

where P is defined by equation (5) and $\boldsymbol{\beta} = (r, \phi_1, \phi_2, \phi_3)$ is the vector of the parameters to be estimated. In more concise form:

³. Longstaff (1990b) also shows how to value options on interest rate "portfolios" (i.e. options on the average of interest rates or on the spread between two interest rates) as well as options on the maximum (or the minimum) of two rates.

$$\mathbf{B}^e = \mathbf{A}\mathbf{P}(\boldsymbol{\tau}, \boldsymbol{\beta}) + \mathbf{u} \quad (54)$$

where $\mathbf{B}^e$ is the vector ($n \times 1$) of the actual (cum-coupon) prices, $\mathbf{A}$ is the payments matrix ($n \times m$) of which the generic element a_{ij} with $i = 1, 2, \dots, n$ e $j=1, 2, \dots, m$ indicates the payment made on the i -th bond at the j -th date, $\mathbf{P}[\boldsymbol{\tau}; \boldsymbol{\beta}]$ is the vector ($m \times 1$) of the coefficients to be estimated and $\mathbf{u}$ is the vector ($n \times 1$) of the errors. It should be noted that the coefficients $\mathbf{P}$ are linked, by way of equation (5), to the vector (known) of the maturities $\mathbf{s}$ and to the vector (unknown) of the parameters $\boldsymbol{\beta}$.

3.2 The score function: least squares

In order to estimate the parameters of model (54) it is necessary to define the score function that is to be minimized, within the predetermined range.

The score function adopted in this paper is the one most commonly used: the sum of squared errors. However, this function is particularly sensitive to the values of outliers, so that a material error in the data (e.g. a decimal point in the wrong place) can seriously distort the results. Accordingly, we modified the estimation procedure by repeating the search for the minimum after automatically excluding the outliers from the data set.

The method involves identifying the residuals that are greater in absolute terms than 2.57 times their standard deviation. On the assumption of normality, the probability of such residuals occurring is one per cent. The observation corresponding to the largest standardized error⁴ in absolute terms is excluded and the search repeated. This procedure is iterated until all the outliers have been eliminated.

3.3 The weighting of the errors

Before carrying out the estimation, we adopted the hypothesis that the variance of the error term $\mathbf{u}$ in (54) increases with the duration of the securities. It is in fact reasonable to assume that operators find it more difficult to determine the value of long-term securities than that of short-term securities, so that the order of magnitude of the errors in the prices of securities closer to maturity will be smaller.

Accordingly, we assumed that the variance of the error is directly proportional to the Hicksian duration (multiplied by the *cum-coupon* price and divided by the return) or, in other words, to the derivative of the price with respect to the yield to maturity with the sign changed. So that:

$$\sigma_u^2 = \frac{\sigma_\epsilon^2 DB_i^e}{(1 + \iota)} \quad (55)$$

$$D = \sum_{j=1}^m \frac{\tau_j a_j}{(1 + \iota)^{\tau_j + 1}} \quad (56)$$

where D is the Hicksian duration and ι is the yield to maturity.

Hence, in order to make the error term homoschedastic, both the members of equation (54) are divided by the square root of DB_i^e , giving:

$$\mathbf{Y} = \mathbf{X}\mathbf{P}(\boldsymbol{\tau}, \boldsymbol{\beta}) + \boldsymbol{\epsilon}. \quad (57)$$

3.4 Marquardt's algorithm

The parameters $\boldsymbol{\beta}$ of model (57) were therefore estimated by minimizing the sum of the squared errors. Denoting this sum as $S(\boldsymbol{\beta})$ we have:

$$\text{Min } S(\boldsymbol{\beta}) = \text{Min } \boldsymbol{\epsilon}'\boldsymbol{\epsilon} \quad (58)$$

⁴. The standardized error is defined as $u/\sqrt{\sum u^2/(n-4)}$.

or, in other terms:

$$\text{Min } S(\beta) = \text{Min} [Y - XP(\tau, \beta)]' [Y - XP(\tau, \beta)] \quad (59)$$

First order conditions require the gradient, i.e. the vector of the four partial derivatives of $S(\beta)$ with respect to β , to be null:

$$z(\tau, \beta)' X' [Y - XP(\tau, \beta)] = 0 \quad (60)$$

where:

$$z(\tau, \beta) = \frac{\partial P(\tau, \beta)}{\partial \beta} = \begin{bmatrix} \frac{\partial P(\tau_1, \beta)}{\partial r} & \frac{\partial P(\tau_1, \beta)}{\partial \phi_1} & \frac{\partial P(\tau_1, \beta)}{\partial \phi_2} & \frac{\partial P(\tau_1, \beta)}{\partial \phi_3} \\ \frac{\partial P(\tau_2, \beta)}{\partial r} & \frac{\partial P(\tau_2, \beta)}{\partial \phi_1} & \frac{\partial P(\tau_2, \beta)}{\partial \phi_2} & \frac{\partial P(\tau_2, \beta)}{\partial \phi_3} \\ \dots & \dots & \dots & \dots \\ \frac{\partial P(\tau_m, \beta)}{\partial r} & \frac{\partial P(\tau_m, \beta)}{\partial \phi_1} & \frac{\partial P(\tau_m, \beta)}{\partial \phi_2} & \frac{\partial P(\tau_m, \beta)}{\partial \phi_3} \end{bmatrix} \quad (61)$$

and the derivatives $\partial P/\partial r$ $\partial P/\partial \phi_1$ $\partial P/\partial \phi_2$ $\partial P/\partial \phi_3$ are defined by (16), (17), (18) and (19).

In order to find the vector $P(\tau; \beta)$ that satisfies equation (60) we used Marquardt's algorithm, which provides, for each stage of the iterative procedure, both the direction matrix and the optimal step length. At the $(k + 1)$ -th iteration the vector β_{k+1} is given by:⁵

$$\beta_{k+1} = \beta_k + [M_k + \lambda_k \text{diag}(M_k)]^{-1} z(\tau, \beta_k)' X' [Y - XP(\tau, \beta_k)] \quad (62)$$

where:

$$M_k = z(\tau, \beta_k)' X' X z(\tau, \beta_k) \quad (63)$$

$\text{diag}(M_k)$ is a matrix with the same elements as M_k along the main diagonal and null elements elsewhere. (64)

For each iteration, if

$$S(\beta_{k+1}) < S(\beta_k) \quad (65)$$

is chosen λ_{k+1} a smaller ($\lambda_{k+1} = \lambda_k/10$) and the iterative procedure is continued.⁶ On the other hand, if

$$S(\beta_{k+1}) > S(\beta_k) \quad (66)$$

a sub-iteration is started in which the vector of the parameters β_k is modified on the basis of equation (62) using an λ_k increasingly large ($\lambda_{k'} = 10\lambda_k$) until inequality (65) holds.⁷

The iterative procedure is continued until the following convergence condition is satisfied:

$$\frac{|S(\beta_{k+1}) - S(\beta_k)|}{S(\beta_k) + \gamma} < \eta \quad (67)$$

where γ and η are arbitrary small numbers.⁸

⁵. The computer program has been subjected to the constraint that the ratio of the values of the β parameters in successive iterations cannot be less than .3 or more than 3. The initial values of the β parameters are, in order: .1117 .25901 .25117 15.617. For k an initial value of .1 has been adopted.

⁶. If λ_k tends towards zero, Marquardt's algorithm converges towards that of Gauss-Newton.

⁷. The maximum number of sub-iterations was set at 10.

3.5 The Gauss-Newton algorithm

Once the parameters of the CIR model have been estimated, in order to value options on pure discount bonds and coupon bonds, it is necessary to determine r^* , the critical rate, corresponding respectively to $P(r^*, T, s) = K$ or $B(r^*, T; \mathbf{s}, \mathbf{a}) = K$.

Since the formulae cannot be explicated in terms of r^* , in either case, we used the Gauss-Newton algorithm, given by:

$$r_{k+1}^* = r_k^* - \frac{B - K}{\frac{\partial B}{\partial r^*}} = r_k^* + \frac{\sum_{j=1}^m a_j P(r_k^*, T, s_j) - K}{\sum_{j=1}^m a_j G(\tau_j) P(r_k^*, T, s_j)} \quad (68)$$

where $\partial B / \partial r^*$ is defined by (16) and (20). The iterative procedure starts with an arbitrary (positive) value r_0 and ends when $|(r_{k+1} - r_k) / r_k| \leq \eta$.

3.6 The finite differences method

In general, when the parameters of the CIR model are known, any financial asset whose price depends exclusively on the current term structure of interest rates can be valued using the finite differences method. For this purpose, the range of (38) in the interval (r, t) is divided into a grid $(n \times m)$ with discrete steps of r equal to h and of t equal to k . The values of $H(r, t)$ are approximated by $H_{ij} = H(ih, jk)$, with $i = 0, 1, 2, \dots, n$ and $j = 0, 1, 2, \dots, m$, and the partial derivatives by the following finite differences:

$$H_r = \frac{H_{i+1,j+1} - H_{i-1,j+1}}{2h} \quad (69)$$

$$H_{rr} = \frac{H_{i+1,j+1} - 2H_{i,j+1} + H_{i-1,j+1}}{h^2} \quad (70)$$

$$H_t = \frac{H_{i,j+1} - H_{i,j}}{k} \quad (71)$$

Substituting equations (69), (70) and (71) in (38) gives the following approximation:

$$H_{i,j} = (aH_{i-1,j+1} + bH_{i,j+1} + cH_{i+1,j+1}) / (1 + rk) \quad i = 1, \dots, n-1 \quad j = 0, \dots, m-1 \quad (72)$$

where

$$a = \left\{ \frac{1}{2} \sigma^2 r / h^2 - \left[\frac{1}{2} \sigma^2 \phi_3 - (2\phi_2 - \phi_1)r \right] / (2h) \right\} k$$

$$b = 1 - \sigma^2 rk / h^2$$

$$c = \left\{ \frac{1}{2} \sigma^2 r / h^2 + \left[\frac{1}{2} \sigma^2 \phi_3 - (2\phi_2 - \phi_1)r \right] / (2h) \right\} k.$$

Equation (72) can be solved iteratively, starting from $H(r, T) = H_{i,m}$.

⁸. The value of η was fixed at .000001 and that of g at .00001. The maximum number of iterations was fixed at 99; in the event of this limit being reached, the parameters β for the iteration with the minimum value of $S(\beta)$ were chosen.

⁹. The initial value r_0 in the computer program was set equal to the current value of the instantaneous interest rate and g to .00001.

Figure 1

4. APPLICATIONS

4.1 Lira, ecu and dollar bonds

The CIR model was estimated independently using three databases referring to fixed rate bonds denominated respectively in:

- lire (BTP), from December 30, 1983 to April 15, 1991;¹⁰
- ecus (CTE), from August 27, 1987 to April 15, 1991;
- dollars (US T-notes and T-bonds), from December 1st, 1983 to January 31, 1991.¹¹

The estimation of the term structures of interest rates using deductive methods (based on a theoretical model) rather than inductive methods (based on simple interpolation) makes it possible to extract considerable information from the prices observed. The CIR model is a useful measurement tool and an effective means of summarizing the information contained in bond prices.¹²

In the first place, it provides historical series of instantaneous and long-term asymptotic interest rates, i.e. of the two extremes of the yield curve, free from the anomalous variations to which the single instruments traded in money and financial markets are subject. The instantaneous and asymptotic rates for lira, ecu and dollar bonds accurately describe the conditions in the markets denominated in these three currencies.

Figure 2

¹⁰. See Barone, Cuoco and Zautzik (1989).

¹¹. See Barone and Salsecci (1991).

¹². An example of the CIR-model's estimation procedure, based on BTPs, is given in Appendix A.

TASSI D'INTERESSE ISTANTANEI IN LIRE E IN DOLLARI:
CROSS-CORRELOGRAMMA DELLE VARIAZIONI GIORNALIERE
(30.12.83 - 31.1.91)

As can be seen in Figure 1, instantaneous interest rates in lire and dollars appear to move in unison. The cross-correlogram of their rates of change shows, however, that the daily changes observed in one market do not help to explain those observed in the other (Figure 2).

Long-term asymptotic interest rates, which reflect both the expectations of operators regarding instantaneous interest rates and the risk premiums (discounts) that they require (grant), are much more variable in Italy than in the United States (Figure 3). Between August 1987 and January 1991, the ratio of the volatility of long-term asymptotic rates to that of instantaneous rates, measured by the standard deviations of their rates of change, was 2.8, 1.8 and 1.0 respectively for lira, ecu and dollar bonds.

The differentials between long and short-term rates summarize the slope of the yield curve (Figure 4). The term structure in lire was downward sloping until June 1986 and then suddenly became upward-sloping; in the United States, the yield curve had a positive slope throughout the period, except for the first half of 1989. In the market for ecu bonds, the yield curve took on a negative slope towards the end of 1989 and kept it for the rest of the period, though it was nearly flat in the last part of 1990. In all three cases the differential widened significantly during 1990.

Figure 3

TITOLI IN LIRE, IN ECU ED IN DOLLARI:
TASSI D'INTERESSE ASINTOTICI A LUNGO TERMINE

Nota: il tasso asintotico a lungo termine e' stato ottenuto stimando i parametri del modello di Cox, Ingersoll e Ross.

Figure 4

TITOLI IN LIRE, IN ECU ED IN DOLLARI:
DIFFERENZIALI TRA TASSI ASINTOTICI A LUNGO TERMINE E TASSI ISTANTANEI

Nota: il tasso istantaneo e il tasso asintotico a lungo termine sono stati ottenuti stimando i parametri del modello di Cox, Ingersoll e Ross.

Figure 5

Implied volatilities reflect operators' uncertainty regarding the development of short-term rates (Figure 5). They became very high at the time of the 1987 stock market crash (October 19), but two years later, at the time of the mini-crash (October 13, 1989), they did not diverge from their normal values.

4.2 (Implied) options on coupon bonds

In Italy there is no official market for bond options..¹³ There are nonetheless securities that embody this type of option and "implicit" prices.

The securities in question are called Treasury option certificates (*Certificati del Tesoro con Opzione* - CTOs), fixed rate bonds with a maturity of six years¹⁴ and the (put) option of early redemption after three years. In other words, they are retractable bonds, which are theoretically equivalent to extendable bonds: in this case, to bonds with a maturity of three years and the (call) option of renewal on the same terms.¹⁵

In particular, the price of CTOs must be above that of both three and six-year bonds. The difference between the price of a CTO and that of a three-year bond is the implied value of the call option, while the difference with respect to the price of a six-year bond is the implied value of the put option.

¹³. There is, however, a fast-growing over-the-counter market on which options on the prices of Treasury bonds and credit certificates are written.

¹⁴. The only exception is the first issue, which had a maturity of 8 years and offered the possibility of early redemption after four years.

¹⁵. See Barone and Cuoco (1989).

In order to assess the extent to which the actual and theoretical values of the options incorporated in these securities coincide, we examined the prices of CTOs in the secondary market (the Milan Stock Exchange and the Wire Market) between June 26, 1989 and April 15, 1991, with 3,804 observations referring to 15 different issues.

The theoretical prices of CTOs were obtained using formulae (20) and (35).¹⁶ The differentials between the actual and theoretical prices have

¹⁶. An example of the estimation procedure for the options implied in CTOs is given in Appendix B.

Table 1

DIFFERENTIAL BETWEEN ACTUAL AND THEORETICAL PRICES OF CTOs

(6/27/89 - 4/15/91)

Code	Security	No. of Observ	Average	Standard Deviation
13043	CTO 1.6.95 12.5%	430	-0.58	0.25
13044	CTO 19.6.95 12.5%	320	-0.60	0.20
13045	CTO 18.7.95 12.5%	296	-0.56	0.22
13049	CTO 16.8.95 12.5%	281	-0.60	0.23
13055	CTO 20.9.95 12.5%	256	-0.56	0.24
13061	CTO 19.10.95 12.5%	229	-0.53	0.28
13065	CTO 20.11.95 12.5%	209	-0.53	0.29
13068	CTO 18.12.95 12.5%	316	-0.39	0.29
13070	CTO 17.1.96 12.5%	179	-0.77	0.20
13073	CTO 19.2.96 12.5%	232	-0.74	0.24
13080	CTO 16.5.96 12.5%	214	-0.69	0.23
13083	CTO 15.6.96 12.5%	160	-0.74	0.20
13086	CTO 19.9.96 12.5%	135	-0.77	0.19
13091	CTO 20.11.96 12.5%	98	-0.70	0.16
13029	CTO 1.12.96 10.25%	449	0.39	0.49
		3,804	-0.58	0.30

a mean of -.58 and a standard deviation of .30 (Table 1). On average, the securities appear to be undervalued.

The size distribution of the differentials is shown in Figure 6. Considered in absolute terms, 38 per cent of the differentials are less than 50 basis points and 93 per cent are less than 100 basis points.

The theoretical values of the call options incorporated in CTOs have a mean of .28, since the probability of the securities being renewed at the early redemption date is very low. This is reflected in a high value of the put option (4.62). One possible explanation is that operators valued CTOs as if the value of the call option was null, or in other words, as if the bonds would in any case be redeemed early.

Figure 7 compares the values of the call and put options implied in the CTOs of the 6/1/95 12.5% issue, while Figure 8 shows their actual and theoretical ex coupon prices. It can be seen that the difference between the latter increases significantly towards the end of the observation period, when interest rates declined sharply. This results in a higher value of the call option, though this does not appear to be reflected in full in the market price of the bonds.

The average undervaluation of CTOs at mid-April 1991 was 98 basis points.

5. CONCLUSIONS

In this paper we describe the techniques needed to value securities whose price depends exclusively on the term structure of interest rates.

Figure 6

DIFFERENZIALI TRA PREZZI EFFETTIVI E TEORICI DEI CTO:
DISTRIBUZIONE PER CLASSI DI AMPIEZZA
(27.6.89 - 15.4.91)

Figure 7

OPZIONI IMPLICITE NEL CTO 1.6.95 12.5%
(27.6.89 - 15.4.91)

Figure 8

Closed-form formulae are presented for the valuation of bonds and call options on bonds and yields, and the put-call parity used to derive the values of put options. We recall that the value of any financial asset whose price depends exclusively on the term structure of interest rates can be obtained using numerical methods (such as the finite differences method used here).

We also describe a method of estimating the parameters of the one-factor CIR model based on non-linear regression techniques (Marquardt's algorithm).

In order to exemplify the method, we applied the model to fixed rate bonds denominated in lire (Italian Treasury bonds - BTPs), ecus (Italian Treasury ecu certificates - CTEs) and dollars (U.S. T-notes and T-bonds) and to Italian Treasury option certificates (CTOs), which are equivalent to combinations of fixed rate bonds and bond options.

APPENDIX A

Estimates of the CIR Model Based on BTPs

Table a1

ESTIMATES OF THE CIR MODEL BASED ON BTPs

(Monday, April 15, 1991)

Code	Security	Tax Rate	Ex-coupon price	Cum-coupon price	Hicksian value	Weight	Theoret	Diffeujhrential
12645	BTP 1.11.91 11,50%	12,5	99,55	104,25	0,52	0,14	104,13	0,11
12648	BTP 21.12.91 11,50%	12,5	99,65	102,95	0,66	0,13	102,69	0,26
12611	BTP 1. 1.92 9,25%	6,25	98,5	101,1	0,69	0,13	101,08	0,02
12613	BTP 1. 2.92 9,25%	6,25	98,6	100,48	0,78	0,12	100,16	0,32
12627	BTP 1. 2.92 11,00%	12,5	99	101,09	0,77	0,12	101,01	0,07
12615	BTP 1. 3.92 9,15%	6,25	98,7	99,84	0,85	0,11	99,26	0,59
12651	BTP 17. 3.92 12,50%	12,5	100	100,97	0,89	0,11	100,84	0,13
12617	BTP 1. 4.92 9,15%	6,25	98,05	98,48	0,94	0,11	98,4	0,08
12631	BTP 1. 4.92 11,00%	12,5	98,8	99,28	0,93	0,11	99,21	0,07
12652	BTP 1. 4.92 12,50% I	12,5	100,05	100,6	0,93	0,11	100,41	0,19
12665	BTP 1. 4.92 12,50% II	12,5	100,05	100,6	0,93	0,11	100,5	0,1
12653	BTP 18. 4.92 12,50%	12,5	100,05	100,18	0,98	0,11	99,9	0,28
12619	BTP 1. 5.92 9,15%	6,25	98,2	102,2	0,98	0,1	101,73	0,48
12634	BTP 1. 5.92 11,00%	12,5	98,6	103,09	0,98	0,1	103	0,09
12654	BTP 1. 5.92 12,50%	12,5	100	105,1	0,97	0,1	104,95	0,16
12655	BTP 17. 5.92 12,50%	12,5	99,9	104,52	1,01	0,1	104,46	0,05
12621	BTP 1. 6.92 9,15%	6,25	97,75	101,04	1,06	0,1	100,9	0,14
12623	BTP 1. 7.92 10,50%	6,25	98,9	101,85	1,14	0,1	101,84	0,01
12638	BTP 1. 7.92 11,50%	12,5	98,85	101,87	1,13	0,1	101,98	-0,11
12640	BTP 1. 8.92 11,50%	12,5	98,65	100,83	1,22	0,09	100,9	-0,07
12642	BTP 1. 9.92 12,50%	12,5	99,9	101,36	1,3	0,09	101,39	-0,03
12644	BTP 1.10.92 12,50%	12,5	99,93	100,48	1,38	0,09	100,5	-0,03
12650	BTP 1. 2.93 12,50%	12,5	99,8	102,17	1,64	0,08	102,3	-0,13
12656	BTP 1. 7.93 12,50%	12,5	99,65	102,93	1,96	0,07	103,01	-0,08
12657	BTP 1. 8.93 12,50%	12,5	99,5	101,87	2,05	0,07	102,01	-0,14
12658	BTP 1. 9.93 12,50%	12,5	99,69	101,15	2,13	0,07	101,09	0,05
12659	BTP 1.10.93 12,50%	12,5	99,85	100,4	2,21	0,07	100,22	0,18
12646	BTP 1.11.93 12,50%	12,5	99,4	104,5	2,18	0,07	104,85	-0,35
12660	BTP 1.11.93 12,50%	12,5	99,4	104,5	2,18	0,07	104,67	-0,17
12647	BTP 17.11.93 12,50%	12,5	99,55	104,17	2,22	0,07	104,49	-0,32
12661	BTP 1.12.93 12,50%	12,5	99,3	103,49	2,26	0,07	103,71	-0,22
12649	BTP 1. 1.94 12,50% I	12,5	99,4	102,68	2,35	0,07	103,01	-0,33
12662	BTP 1. 1.94 12,50% II	12,5	99,38	102,66	2,35	0,07	102,71	-0,05
12663	BTP 1. 2.94 12,50%	12,5	99,2	101,57	2,43	0,07	101,73	-0,16
12664	BTP 1. 3.94 12,50%	12,5	99,37	100,83	2,51	0,07	100,87	-0,04
12666	BTP 1. 5.94 12,50%	12,5	99,15	104,25	2,54	0,06	104,45	-0,2
12667	BTP 1. 6.94 12,50%	12,5	99,1	103,29	2,62	0,06	103,62	-0,33
12670	BTP 1. 7.94 12,50%	12,5	99,12	102,4	2,7	0,06	102,75	-0,35
12671	BTP 1. 9.94 12,50%	12,5	99,09	100,55	2,87	0,06	100,73	-0,19
12672	BTP 1.11.94 12,50%	12,5	99,08	104,18	2,89	0,06	104,31	-0,12
12674	BTP 1. 1.96 12,50%	12,5	98,56	101,84	3,68	0,05	102	-0,16
12676	BTP 1. 3.96 12,50%	12,5	98,55	100,01	3,85	0,05	100,17	-0,17
12668	BTP 1. 6.97 12,50%	12,5	98,05	102,24	4,42	0,05	102,34	-0,1
12669	BTP 16. 6.97 12,50%	12,5	98,27	102,01	4,47	0,05	102,05	-0,04
12673	BTP 1.11.97 12,50%	12,5	98,13	103,23	4,59	0,05	103,05	0,18
12675	BTP 1. 1.98 12,50%	12,5	98,29	101,57	4,76	0,05	101,17	0,4
12678	BTP 19. 3.98 12,50%	12,5	98,19	99,1	4,97	0,05	98,8	0,3
12677	BTP 1. 3. 1 12,50%	12,5	97,64	99,1	6,14	0,04	98,37	0,73

N.B.: the estimates of the parameters b of the Cox, Ingersoll and Ross model are .10266 .25897 25124 15.016..

Table a2

ELEMENTS OF MATRIX A

(i=1,...,48 and j=1,111)

i	$a_{i,j}$			
1.00	5,0313	(j=1)	104,92	(j=13)
2.00	5,0313	(j=5)	104,94	(j=17)
3.00	4,3359	(j=6)	104,26	(j=18)
4.00	4,3359	(j=7)	104,26	(j=19)
5.00	4,8125	(j=7)	104,69	(j=19)
6.00	4,2891	(j=8)	104,21	(j=20)
7.00	5,4688	(j=9)	105,19	(j=21)
8.00	4,2891	(j=11)	104,21	(j=23)
9.00	4,8125	(j=11)	104,56	(j=23)
10.00	5,4688	(j=11)	105,19	(j=23)
11.00	5,4688	(j=11)	105,29	(j=23)
12.00	5,4688	(j=12)	105,27	(j=24)
13.00	4,2891	(j=1,13)	104,21	(j=25)
14.00	4,8125	(j=1,13)	104,5	(j=25)
15.00	5,4688	(j=1,13)	105,25	(j=25)
16.00	5,4688	(j=2,14)	105,19	(j=26)
17.00	4,2891	(j=3,15)	104,21	(j=27)
18.00	4,9219	(j=6,18)	104,91	(j=29)
19.00	5,0313	(j=6,18)	104,83	(j=29)
20.00	5,0313	(j=7,19)	104,71	(j=30)
21.00	5,4688	(j=8,20)	105,32	(j=31)
22.00	5,4688	(j=11,23)	105,32	(j=33)
23.00	5,4688	(j=7,19,30)	105,41	(j=39)
24.00.00	5,4688	(j=6,18,29,38)	105,19	(j=47)
25.00.00	5,4688	(j=7,19,30,39)	105,14	(j=48)
26.00.00	5,4688	(j=8,20,31,40)	105,13	(j=49)
27.00.00	5,4688	(j=11,23,33,42)	105,17	(j=51)
28.00.00	5,4688	(j=1,13,25,34,43)	105,36	(j=52)
29.00.00	5,4688	(j=1,13,25,34,43)	105,12	(j=52)
30.00.00	5,4688	(j=2,14,26,35,44)	105,49	(j=53)
31.00.00	5,4688	(j=3,15,27,36,45)	105,05	(j=54)
32.00.00	5,4688	(j=6,18,29,38,47)	105,44	(j=56)
33.00.00	5,4688	(j=6,18,29,38,47)	105,05	(j=56)
34.00.00	5,4688	(j=7,19,30,39,48)	104,95	(j=57)
35.00.00	5,4688	(j=8,20,31,40,49)	104,95	(j=58)
36.00.00	5,4688	(j=1,13,25,34,43,52)	104,95	(j=60)
37.00.00	5,4688	(j=3,15,27,36,45,54)	105,09	(j=61)
38.00.00	5,4688	(j=6,18,29,38,47,56)	105,18	(j=63)
39.00.00	5,4688	(j=8,20,31,40,49,58)	105	(j=64)
40.00.00	5,4688	(j=1,13,25,34,43,52,60)	105	(j=66)
41.00.00	5,4688	(j=6,18,29,38,47,56,63,69,75)	104,89	(j=81)
42.00.00	5,4688	(j=8,20,31,40,49,58,64,70,76)	104,89	(j=82)
43.00.00	5,4688	(j=3,15,27,36,45,54,61,67,73,79,85,91)	104,72	(j=97)
44.00.00	5,4688	(j=4,16,28,37,46,55,62,68,74,80,86,92)	105,02	(j=98)
45.00.00	5,4688	(j=1,13,25,34,43,52,60,66,72,78,84,90,96)	104,66	(j=102)
46.00.00	5,4688	(j=6,18,29,38,47,56,63,69,75,81,87,93,99)	104,66	(j=103)
47.00.00	5,4688	(j=10,22,32,41,50,59,65,71,77,83,89,95,101)	104,66	(j=105)
48.00.00	5,4688	(j=8,20,31,40,49,58,64,70,76,82,88,94,100,104,106,107,108,109,110)	104,49	(j=111)

Table a3

VECTORS $\mathbf{s}$ τ and $P(\tau)$ $(j = 1, 111)$

j	τ_j	$P(\tau_j)$	j	τ_j	$P(\tau_j)$	j	τ_j	$P(\tau_j)$
1	14	0,99607	38	627	0,83401	75	1537	0,63322
2	29	0,99186	39	655	0,82712	76	1597	0,62163
3	46	0,98711	40	683	0,82026	77	1615	0,61819
4	60	0,98321	41	701	0,81588	78	1659	0,60986
5	64	0,98210	42	714	0,81273	79	1688	0,60442
6	74	0,97932	43	746	0,80502	80	1705	0,60126
7	105	0,97074	44	760	0,80166	81	1720	0,59848
8	137	0,96193	45	775	0,79808	82	1779	0,58766
9	152	0,95782	46	790	0,79451	83	1797	0,58439
10	154	0,95727	47	805	0,79095	84	1841	0,57648
11	166	0,95399	48	837	0,78341	85	1873	0,57079
12	183	0,94936	49	867	0,77639	86	1887	0,56832
13	200	0,94474	50	886	0,77197	87	1901	0,56586
14	214	0,94095	51	897	0,76942	88	1964	0,55490
15	228	0,93717	52	929	0,76205	89	1981	0,55198
16	242	0,93340	53	944	0,75862	90	2027	0,54414
17	249	0,93152	54	958	0,75542	91	2055	0,53942
18	259	0,92884	55	973	0,75201	92	2069	0,53708
19	291	0,92029	56	991	0,74794	93	2086	0,53425
20	319	0,91286	57	1020	0,74142	94	2146	0,52436
21	334	0,90889	58	1048	0,73516	95	2162	0,52175
22	336	0,90836	59	1068	0,73073	96	2206	0,51464
23	349	0,90494	60	1110	0,72148	97	2237	0,50969
24	369	0,89969	61	1140	0,71494	98	2251	0,50746
25	382	0,89629	62	1155	0,71169	99	2266	0,50509
26	396	0,89264	63	1170	0,70845	100	2328	0,49541
27	410	0,88900	64	1232	0,69521	101	2346	0,49263
28	425	0,88511	65	1250	0,69141	102	2391	0,48575
29	440	0,88124	66	1294	0,68218	103	2451	0,47671
30	473	0,87276	67	1323	0,67617	104	2510	0,46799
31	502	0,86536	68	1338	0,67307	105	2527	0,46550
32	522	0,86028	69	1355	0,66958	106	2693	0,44189
33	532	0,85775	70	1413	0,65779	107	2874	0,41747
34	564	0,84970	71	1432	0,65397	108	3058	0,39398
35	579	0,84594	72	1475	0,64540	109	3240	0,37202
36	593	0,84245	73	1505	0,63948	110	3424	0,35103
37	608	0,83872	74	1520	0,63654	111	3605	0,33153

Figure a1

Figure a2

Figure a3

Figure a4

Figure a5

Figure a6

APPENDIX B

Estimates of the Options Implied in CTOs

Table b1

CTOs QUOTED ON THE SECONDARY MARKET										
(Monday, April 15, 1991)										
Code	Security	Tax Rate	Ex- Coupon Price	Cum- Coupon Price	Theoret value	Theoret Diffeential	Maturity ^a	Critical Rate	Call	Put
13043	CTO 1. 6.95 12,50%	12,5	100,07	104,26	105.14	-0,88	410	10,114	0,85	1,71
13044	CTO 19. 6.95 12,50%	12,5	99,95	103,6	104.62	-1,03	428	10,143	0,88	1,71
13045	CTO 18. 7.95 12,50%	12,5	99,95	102,71	103.68	-0,97	459	10,123	0,87	1,76
13049	CTO 16. 8.95 12,50%	12,5	99,8	101,71	102.86	-1,14	487	10,134	0,89	1,78
13055	CTO 20. 9.95 12,50%	12,5	99,85	100,73	101.81	-1,08	522	10,117	0,88	1,83
13061	CTO 19.10.95 12,50%	12,5	99,85	105,32	106.43	-1,12	550	10,097	0,87	1,87
13065	CTO 20.11.95 12,50%	12,5	99,85	104,38	105.42	-1,05	582	10,045	0,84	1,95
13068	CTO 18.12.95 12,50%	12,5	99,96	103,64	104.57	-0,94	610	10,056	0,85	1,96
13070	CTO 17. 1.96 12,50%	12,5	99,5	102,65	103.64	-1,00	641	10,071	0,86	1,96
13073	CTO 19. 2.96 12,50%	12,5	99,93	101,75	102.66	-0,90	673	10,045	0,84	2,01
13080	CTO 16. 5.96 12,50%	12,5	99,91	104,56	105.47	-0,91	760	10,046	0,84	2,05
13083	CTO 15. 6.96 12,50%	12,5	99,82	103,59	104.58	-0,99	789	10,061	0,85	2,05
13086	CTO 19. 9.96 12,50%	12,5	99,84	100,75	101.63	-0,87	886	10,064	0,84	2,06
13091	CTO 20.11.96 12,50%	12,5	99,83	104,36	105.17	-0,81	949	10,079	0,84	2,06
13029	CTO 1.12.96 10,25%	12,5	96,25	99,69	100.67	-0,98	593	6,940	0,02	6,91

(a) Days.

N.B.: the estimates of the parameters b of the Cox, Ingersoll and Ross model are .10266 .25897 .25124 15.016.

Table b2

ELEMENTS OF MATRIX A				
(i=1,...,15 and j=1,122)				
i	a _{i,j}			
1	5,4688	(j=4,17,30,42,54,65,76,87)	105,29	(j=98)
2	5,4688	(j=7,20,33,45,57,67,78,89)	105,38	(j=100)
3	5,4688	(j=9,22,35,46,58,69,79,91)	105,24	(j=102)
4	5,4688	(j=10,23,36,47,59,70,80,92)	105,31	(j=103)
5	5,4688	(j=13,26,38,50,61,72,83,94)	105,26	(j=106)
6	5,4688	(j=1,14,27,39,51,62,73,84,95)	105,24	(j=107)
7	5,4688	(j=3,16,29,41,53,64,75,86,97)	105,11	(j=109)
8	5,4688	(j=6,19,32,44,56,67,78,89,100)	105,15	(j=112)
9	5,4688	(j=8,21,34,46,58,68,79,90,101)	105,14	(j=113)
10	5,4688	(j=11,24,37,48,60,71,81,93,104)	105,11	(j=114)
11	5,4688	(j=2,15,28,40,52,63,74,85,96,108)	105,11	(j=116)
12	5,4688	(j=5,18,31,43,55,66,77,88,99,111)	105,25	(j=119)
13	5,4688	(j=12,25,38,49,61,72,82,94,105,115)	105,15	(j=120)
14	5,4688	(j=3,16,29,41,53,64,75,86,97,109,117)	105,15	(j=121)
15	4,4844	(j=4,17,30,42,54,65,76,87,98,110,118)	103,86	(j=122)

Table b3

VECTORS s and $P(s)$								
(j=1,122))								
j	s_j	$P(s_j)$	j	s_j	$P(s_j)$	j	s_j	$P(s_j)$
1	1	0,99972	42	593	0,84245	83	1251	0,69119
2	28	0,99214	43	607	0,83897	84	1280	0,68511
3	32	0,99102	44	610	0,83822	85	1308	0,67927
4	46	0,98711	45	613	0,83748	86	1313	0,67824
5	60	0,98321	46	641	0,83056	87	1323	0,67617
6	61	0,98293	47	670	0,82344	88	1337	0,67328
7	62	0,98266	48	673	0,82271	89	1341	0,67246
8	90	0,97488	49	701	0,81588	90	1370	0,66652
9	91	0,97461	50	704	0,81516	91	1371	0,66631
10	120	0,96660	51	732	0,80839	92	1400	0,66042
11	123	0,96578	52	760	0,80166	93	1404	0,65961
12	154	0,95727	53	763	0,80094	94	1432	0,65397
13	155	0,95700	54	775	0,79808	95	1462	0,64798
14	186	0,94854	55	789	0,79475	96	1489	0,64263
15	214	0,94095	56	792	0,79403	97	1495	0,64145
16	216	0,94041	57	795	0,79332	98	1505	0,63948
17	228	0,93717	58	823	0,78670	99	1519	0,63673
18	242	0,93340	59	851	0,78012	100	1523	0,63595
19	244	0,93286	60	854	0,77942	101	1551	0,63050
20	245	0,93259	61	886	0,77197	102	1552	0,63030
21	274	0,92482	62	915	0,76527	103	1581	0,62470
22	277	0,92402	63	943	0,75885	104	1586	0,62374
23	305	0,91657	64	949	0,75748	105	1615	0,61819
24	307	0,91604	65	958	0,75542	106	1616	0,61800
25	336	0,90836	66	972	0,75224	107	1645	0,61250
26	337	0,90810	67	977	0,75111	108	1673	0,60723
27	369	0,89969	68	1005	0,74478	109	1677	0,60648
28	396	0,89264	69	1006	0,74456	110	1688	0,60442
29	398	0,89212	70	1035	0,73806	111	1702	0,60182
30	410	0,88900	71	1040	0,73695	112	1705	0,60126
31	424	0,88537	72	1068	0,73073	113	1735	0,59571
32	427	0,88459	73	1097	0,72433	114	1768	0,58966
33	428	0,88434	74	1124	0,71842	115	1797	0,58439
34	456	0,87712	75	1128	0,71755	116	1855	0,57399
35	459	0,87635	76	1140	0,71494	117	1859	0,57328
36	487	0,86918	77	1154	0,71191	118	1873	0,57079
37	489	0,86867	78	1159	0,71083	119	1887	0,56832
38	522	0,86028	79	1187	0,70480	120	1981	0,55198
39	550	0,85321	80	1216	0,69861	121	2043	0,54144
40	578	0,84619	81	1219	0,69797	122	2055	0,53942
41	582	0,84519	82	1250	0,69141			

Table b4

CTO 6/1/95 12.5%: THEORETICAL VALUE OF CALL OPTION

(formula [35])

j	a _j	P(s _j)	d _{1,j}	df _{1,j}	nc _{1,j}	v ² (d _{1,j} ...)	K	P(T)	d ₂	df ₂	nc ₂	v ² (d ₂ ...)	c _j
1	5,4688	0,84245	106,03	30	81,54	0,40696	100	0,889	105,94	30	81,614	0,40358	0,001987
2	5,4688	0,79808	106,11	30	81,476	0,40994	100	0,889	105,94	30	81,614	0,40358	0,003561
3	5,4688	0,75542	106,19	30	81,418	0,41259	100	0,889	105,94	30	81,614	0,40358	0,004798
4	5,4688	0,7149	106,25	30	81,368	0,41492	100	0,889	105,94	30	81,614	0,40358	0,00574
5	5,4688	0,67617	106,31	30	81,324	0,41699	100	0,889	105,94	30	81,614	0,40358	0,006444
6	105,29	0,63948	106,36	30	81,285	0,41881	100	0,889	105,94	30	81,614	0,40358	0,006943

$$c = \sum_{j=1}^6 a_j c_j = 0,85426.$$

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