**THE TERM STRUCTURE OF INTEREST RATES:
A TEST OF THE COX, INGERSOLL AND ROSS MODEL
ON ITALIAN TREASURY BONDS^(*)**

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SUMMARY

This paper tests the Cox, Ingersoll and Ross model using the prices of Italian Treasury bonds in the secondary market. The model is estimated daily for the period 30 December 1983 to 13 March 1989. The resulting term structures of interest rates are compared with those obtained using interpolation techniques (the cubic splines method). The daily estimation of the yield curves also makes it possible to analyze the changes in Treasury bond prices, determine the turning points and obtain useful indications regarding the efficiency of the secondary market and the consistency between the primary and the secondary markets.

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1. INTRODUCTION

The conditions in the bond market at any given moment can be summarized by the yield curve:¹ in a perfect market without taxes, arbitrage operations ensure that the prices of default-free bonds are equal to the present value of future payments (in the form of interest and principal), with each being discounted at the appropriate rate (depending on the date of payment)². In other words,

$$B(t, t_n) = \sum_{i=1}^n c_i e^{-R(t, t_i)(t_i - t)} = \sum_{i=1}^n c_i P(t, t_i) \quad (1)$$

where:

$B(t, t_n)$ is the price at time t of a coupon bond maturing at t_n

c_i is the payment made at time t_i ($t < t_i < t_n$)

$R(t, t_i)$ is the interest rate for the maturity t_i

$P(t, t_i)$ is the price of a zero-coupon bond maturing at time t_i .

The yield curve expresses the relationship at time t between the rates of return $R(t, T)$ on zero-coupon bonds and their maturity T . It can be obtained by transforming the function that links the prices $P(t, T)$ of pure discount bonds to their maturity (the discount function), given that:

$$R(t, T) = \frac{-\ln[P(t, T)]}{T - t} \quad (2)$$

The problems encountered in estimating the yield curve in Italy derive from the absence of zero-coupon bonds with maturities of more than one year in the market for government securities and from the consequent difficulty of determining, on the basis of coupon bond prices, the factors to be used to discount payments beyond the short term³.

One simple procedure involves treating coupon bonds as zero-coupon bonds with a maturity equal to their duration⁴. Using this method, the yield curve is accordingly estimated by regressing the yields to maturity of coupon bonds on their duration. However, this approach is open to criticism on theoretical grounds⁵.

Alternatively, equation (1) can be used to obtain the prices of zero-coupon bonds (P) from those of coupon bonds (B). Taking the vector $\mathbf{b}$ of the n observed prices of coupon bonds as the dependent variable and the matrix $\mathbf{C}$ of the $(n \times m)$ payments as the independent variable, multiple regression techniques can be applied to the equation

$$\mathbf{b} = \mathbf{C}\mathbf{p} + \boldsymbol{\varepsilon} \quad (3)$$

to obtain an estimate of the vector $\mathbf{p}$ of the m coefficients. The error $\boldsymbol{\varepsilon}$ in (3) reflects the existence of transaction costs, taxation, the asynchrony of the observed prices and other market imperfections.

In order to estimate (3) using ordinary regression techniques, the number of observed prices (n) must be at least equal to the number of payment dates (m). Since n is normally less than m , it can be

¹ The term structure of interest rates in the Italian bond market was first analyzed by R.S. Masera (1972).

² With transaction costs and taxes, which can have a different effect on different securities and holders, there is a multiplicity of term structures. Cf. J.C. Dermody and E.Z. Prisman (1988).

³ Since (1) is a linear function of prices $P(t, t_i)$, it is generally better to use this equation to estimate the discount function rather than having recourse directly to the term structure.

⁴ This procedure was originally proposed by Macaulay (1938).

⁵ See M. Livingston and J. Caks (1977).

assumed that payments are made only in correspondence with a smaller set of dates⁶. To prevent the forward rates implicit in the curve being negative, it is also necessary to impose the condition that prices (P) are monotonically not increasing with respect to the maturity and fall between 0 and 1⁷. However, this approach gives a discrete (rather than continuous) yield curve that may be extremely irregular.

An alternative method is to impose the condition that the prices of pure discount bonds in (1) are related to maturities by a continuous function. The form of this function can be suggested by the actual figures (the inductive approach) or specified in advance on the basis of an equilibrium model (the deductive approach).

Among the deductive models, that developed by Cox, Ingersoll and Ross (CIR) provides a complete specification, in a general equilibrium setting, of the term structure of zero-coupon bond prices⁸. In its simplest form the CIR model is based on the following assumptions: (1) the short-term interest rate moves randomly around a path that leads towards its long-term equilibrium level and (2) the rates of return expected in the short term on investments in bonds of any maturity are equal to the short-term interest rate plus a risk premium.

The rest of the paper is divided into three sections. Section 2 contains a description of the CIR model. In Section 3 we compare the curves obtained with the model with those obtained using an inductive approach based on cubic splines. In both cases the parameters are estimated on the prices of Italian Treasury bonds listed on the secondary market (Milan Stock Exchange and the Wire Market). The CIR model is also used to make a retrospective analysis of the issue conditions of Treasury bills and bonds during the period in question. This analysis provides indications regarding the efficiency of the secondary market and its consistency *vis-a-vis* the primary market. Section 4 contains the main conclusions.

2. THE COX, INGERSOLL AND ROSS MODEL

2.1 Description of the model

The one-factor formulation of the CIR model assumes that:

1. the instantaneous interest rate (r) follows the process described by the differential stochastic equation

$$dr = \kappa(\mu - r)dt + \sigma\sqrt{r}dz \quad (4)$$

where κ is the speed of adjustment of the interest rate r towards its long-run expected value μ , $\sigma\sqrt{r}$ is the volatility of the variations in r and dz is a standard Wiener process. If $0 < \kappa < \infty$ the instantaneous rate tends to the average value μ (*mean-reverting process*);

2. the expected instantaneous rate of return on bonds of any maturity is equal to the interest rate r plus a risk premium (*Local Expectations Hypothesis*)

$$E\left(\frac{dB}{B}\right) = \left(r + \lambda r \frac{B_r}{B}\right) \quad (5)$$

where $-\lambda$ is the market price of risk and rB_r/B is the elasticity of the price (B) of the bond with respect to r ;

3. the usual perfect market assumptions apply.

⁶ In this case individual payments are “shifted” to one of the dates in the chosen set using, for example, the yield to maturity of the security.

⁷ The imposition of constraints on the coefficients in the form of linear inequalities turns the regression into a problem of quadratic programming - see W.R. Carleton and I. Cooper.

⁸ See J.C. Cox, J.E. Ingersoll and S.A. Ross (1985 a, b).

Given these assumptions, it can be shown that the value $H = H(r, t, T)$ of any financial asset depending exclusively on the current term structure of interest rates varies (in the interval between two successive coupons) in accordance with the differential stochastic equation

$$\frac{1}{2}\sigma^2 r H_{rr} + [\kappa(\mu - r) - \lambda r] H_r + H_t - rH = 0. \quad (6)$$

This equation can be solved once the appropriate boundary conditions have been imposed in the light of the features of each financial asset.

For the price $P(t, T)$ of a pure discount bond with a residual life $\tau = T - t$, the solution of equation (6), subject to the boundary condition

$$P(T, T) = 1 \quad (7)$$

is

$$P(t, T) = F(t, T) e^{-G(t, T)r} \quad (8)$$

where

$$F(t, T) = \left[\frac{\phi_1 e^{\phi_2 \tau}}{\phi_2 (e^{\phi_1 \tau} - 1) + \phi_1} \right]^{\phi_3}$$

and

$$G(t, T) = \left[\frac{(e^{\phi_1 \tau} - 1)}{\phi_2 (e^{\phi_1 \tau} - 1) + \phi_1} \right].$$

Equation (8) expresses the value of a pure discount bond of maturity τ as a function of the state variable r and of the three parameters ϕ_1 , ϕ_2 and ϕ_3 , where:

$$\phi_1 = \sqrt{[\kappa + \lambda]^2 + 2\sigma^2} \quad (9)$$

$$\phi_2 = \frac{(\kappa + \lambda + \phi_1)}{2} \quad (10)$$

$$\phi_3 = \frac{2\kappa\mu}{\sigma^2}. \quad (11)$$

Equation (8) enables the whole term structure $R(t, T)$ to be determined using (2). In particular:

$$\lim_{T \rightarrow 0} R(t, T) = r \quad (12)$$

$$\lim_{T \rightarrow \infty} R(t, T) = (\phi_1 - \phi_2)\phi_3. \quad (13)$$

The parameter σ of (4) is given by the equation:

$$\sigma = \sqrt{2(\phi_1 \phi_2 - \phi_2^2)}. \quad (14)$$

In this model the basis risk associated with a pure discount bond, i.e. the proportional variation in its price produced by an infinitesimal change in the instantaneous interest rate (with the sign changed), is given by (8) as:

$$\frac{-P_r}{P} = G(\tau). \quad (15)$$

The basis risk of a coupon bond is therefore equal to the weighted average of the basis risks of the discount bonds with maturities at the various payment dates, with weights equal to the present values of the payments:

$$\frac{-B_r}{B} = \frac{-\sum cP_r}{\sum cP} = \frac{\sum cPG(\tau)}{\sum cP} \quad (16)$$

This measure of risk for coupon bonds can be expressed in units of time; in this case it has been called stochastic duration by Cox, Ingersoll and Ross. The stochastic duration (τ) of a coupon bond is equal to the maturity of the discount bond with the same level of risk:

$$\tau = G^{-1}\left[\frac{Br}{B}\right] = \frac{1}{\phi_1} \ln\left[1 - \frac{B_r / B\phi_1}{1 + B_r / B\phi_2}\right]. \quad (17)$$

2.2 Estimation methods

The parameters ϕ_1 , ϕ_2 and ϕ_3 of the model and the state variable r can be estimated by applying a non-linear regression technique or the generalized method of moments to equation (3). However, the method of moments is difficult to apply in the case of coupon bonds. Accordingly, the parameters of the CIR model have been obtained here by means of non-linear regression (Marquardt's method)⁹.

When estimating the model, it was assumed that the residuals of (2) were mutually independent and normally distributed, with a zero mean and a variance proportional to the derivatives of the bond prices with respect to their yields to maturity (in other words, to the products of their durations and cum coupon prices). In fact, it appears reasonable to expect the errors in the prices of bonds with very short maturities to be smaller than those for bonds with long maturities. Accordingly, the coefficients were estimated using generalized least squares, dividing both members of equation (2) by the square root of the derivatives.

In addition, the following iterative procedure was adopted to exclude outliers: if any observation had a residual greater, in absolute value, than 2.57 times the standard deviation (corresponding to a probability of 0.01), the observation with the largest standardized residual was excluded and the regression re-estimated. The procedure was repeated until all the outliers were eliminated.

In the hypotheses underlying the model, the parameters ϕ_1 , ϕ_2 and ϕ_3 , are constant. It might appear inefficient or even inconsistent to estimate these parameters on the basis of successive cross sections, without imposing intertemporal constraints on the estimates. The same argument applies to the parameter s expressed by (14).

In our estimates we did not impose any constraints on the estimated parameters. The constancy of the parameters assumed in the model implies that the rate of return of a long-term zero-coupon bond is constant, which appears appropriate for the real yield curve but not for the nominal yield curve. Furthermore, the behaviour of the time-series of the parameters of the model, especially the implied volatility, is of interest in its own right.¹⁰

⁹ For estimations using non-linear regression techniques, see S. Brown and P. Dybvig (1986) and R. Brown and S. Schaefer (1988). For an estimation based on the method of moments, see M. Gibbons and K. Ramaswamy (1986).

An alternative method, known as the two-stage method, consists in first estimating the parameters κ , μ and σ of process (4) followed by the instantaneous interest rate, using the time series of a short-term rate, and then λ on the basis of equation (3): see A. Ananthanarayanan and E. Schwartz (1980). For an application of the two-stage method to the Italian market, see E. Barone and R. Cesari (1986).

¹⁰ R.H. Brown and S.M. Schaefer (1988) stress the similarity between this type of approach and the calculation, based on the formula of Black and Scholes, of the implied standard deviation (ISD) in the prices of options. "Strictly speaking, intertemporal variation in the ISD will be almost always inconsistent with the model. Moreover, to the extent that actual volatility changes over time, the assumptions of the Black-Scholes model are often violated. Nonetheless, the model is found to explain the cross-sectional structure of option prices well, and ISD's are found to be good predictors of volatility".

3. RESULTS

3.1 Data

The term structure was estimated using the daily prices of the Treasury bonds traded on the Milan Stock Exchange and the Wire Market from 30 December 1983 to 13 March 1989.¹¹ For the bonds listed on both markets, the Wire Market quotations were used, with the average of the best bid and ask prices at 4 p.m. being used when a bond was not actually traded. At the beginning of the period 7 bonds were observed with a face value of 15,890 billion lire, while at the end of 1988 these figures had risen to respectively 47 and 112,527 billion (Table 1).¹²

The introduction of the withholding tax on the interest on government bonds in September 1986 is a complication.¹³ Since corporate bodies tend to determine the profitability of securities in terms of their gross yields while individuals base their calculations on net yields, it is possible to establish a priori that the “true” equilibrium market rate will lie somewhere in between, but it is not possible to determine exactly where.

Net rates have been used in what follows since the bulk of government securities are held by households, making it reasonable to assume that the equilibrium rate is more directly approximated by the net yield.

3.2 Estimation of the CIR model

The CIR model was estimated on a daily basis for the period from 30 January 1983 to 13 March 1989.

Some of the figures regarding the time-series of the parameters ϕ_1 , ϕ_2 and ϕ_3 are given in Table 2. The stability of the estimates increased in recent years (with the notable exception of 1987), when the number of securities observed was larger.

Figure 1 shows the distribution of the differentials between the actual and theoretical prices for the whole sample (26,058 observations) and the normal distribution with the same mean and variance as the sample. The mean of the distribution amounted to 0.04, which is not significantly different from zero (the t statistic was equal to 0.07). In 25 per cent of the cases the absolute value of the differential was less than 0.10 and in 80 per cent it was less than 0.50. The variance of the actual prices is almost entirely explained by the model ($R^2 \approx 1$).

In general the differentials are smaller for the shorter maturities and tend to become larger for longer maturities.¹⁴ The small discrepancies found among the shorter maturities can be attributed to the existence of transaction costs, which make arbitrage operations unprofitable when the price differential is small. The large differences sometimes found among the longer maturities can be attributed, instead, to the existence of securities and operators subject to different tax arrangements.¹⁵

On average the theoretical prices given by the model are very close to those recorded in the market. However, in view of the method used to estimate the parameters, this is not in itself sufficient to ensure the model is economically significant and that the procedure adopted is not simply a

¹¹ Treasury bonds are traded “spot”, i.e. for settlement on the third working day subsequent to the day the contract is entered into. Prices are customarily quoted ex coupon; cum coupon prices are obtained by summing the ex coupon price and the after-tax amount of the current coupon maturing (using simple interest and the commercial year) in the period between the day accrual starts and the settlement day (inclusive of both).

¹² The tables are shown in Appendix B.

¹³ The withholding tax frees individuals from all further fiscal obligations with regard to interest on government bonds, which does not have to be included in their income tax returns. Corporate bodies are required to include interest income in their returns and receive a tax credit for the withholding tax already paid.

¹⁴ These empirical results confirm the specification of the vector of the residuals e in estimating the parameters of the model (cf. §2.2).

¹⁵ The market shows a slight tendency (statistically insignificant) to overvalue securities subject to withholding tax at the rate of 6.25 per cent and to undervalue those taxed at the rate of 12.50 per cent. The mean difference in the first case is equal to 0.18 and to -0.12 in the second. This may be connected with the problem of determining the relevant yield (net or gross): cf. footnote 2.

Figure 1

sophisticated method of interpolating the data. Additional tests are therefore required to ascertain the consistency of the estimated values of the economically significant variables.

Table 3 gives some figures regarding the estimated time-series of the state variable r (the instantaneous interest rate) and of the long-term asymptotic rate obtained using equation (13). The correlation coefficients between the instantaneous rate and selected domestic and foreign money market rates are shown in Table 4. The correlation is particularly high (0.98) for 3-month Treasury bills. Figure 2 compares the instantaneous rate and the net yield on 3-month Treasury bills.

The results of the model are consistent with a-priori expectations. The short-term rate falls sharply between 1984 and 1986 and remains upward basically stable from the end of that year on. The long-term rate is more variable, especially in the earlier years. It is nonetheless possible to distinguish a clear downward trend up to the middle of 1985 and a slight trend, albeit with large oscillations in both directions, from the end of 1985 on (Figure 3).

In the middle of 1986 the long-term rate overtook the short-term rate, causing the slope of the yield curve to change from negative, in line with the prevalence of expectations of a further fall in interest rates, to positive (Figure 4). The change was extremely sharp and the daily figures show that it occurred in the last week of June.¹⁶

Further evidence regarding the reliability of the model can be obtained by examining the volatility of the instantaneous interest rate implicit in the estimated yield curves ($\sigma\sqrt{r}$).¹⁷

La Table 5 shows some figures regarding the time-series of the implied volatility. Starting in 1987, the values are very close to those obtained using the time-series of the instantaneous rate

¹⁶ This tallies with the perception of the situation at the time as reported in the 1986 Annual Report of the Bank of Italy: "Starting in the summer the control of the monetary variables became more difficult as a result of the market's irregular demand for government securities, which reflected the weakening of expectations of further reductions in interest rates, especially in July".

¹⁷ The volatility of the instantaneous rate is defined as the standard deviation of the absolute variations: cf. (4).

(Table 3), and this is an important test of the CIR model since the implied volatility is calculated on the basis of a cross-section of securities.

Figure 2

INSTANTANEOUS INTEREST RATE AND NET YIELD ON 3-MONTH TREASURY BILLS
(Raw data: Italian Treasury bond prices)

N.B. The instantaneous rate was obtained by estimating the parameters of the Cox, Ingersoll and Ross model.

Figure 3

INSTANTANEOUS AND LONG-TERM ASYMPTOTIC INTEREST RATES
(Raw data: Italian Treasury bond prices)

N.B. The instantaneous rate (r) and the long-term asymptotic rate (r_1) were obtained by estimating the parameters of the Cox, Ingersoll and Ross model.

Figure 4

DIFFERENTIAL BETWEEN LONG-TERM ASYMPTOTIC AND INSTANTANEOUS INTEREST RATES
AND IMPLIED VOLATILITY OF THE INSTANTANEOUS RATE
(Raw data: Italian Treasury bond prices)

N.B. The instantaneous interest rate, the long-term asymptotic rate and the implied volatility were obtained by estimating the parameters of the Cox, Ingersoll and Ross model.

Between 1984 and 1985 the implied volatility is found to be quite unstable and decreasing (from around 8 per cent to around 5 per cent); subsequently, it is generally more stable and increases slightly (Figure 4). The greater stability of this parameter may be partly attributable to the gradual development of the securities market in recent years.

It is worth noting that the much greater volatility lasting from the middle of October 1987 and the beginning of 1988 was the only break of any length in the recent period of stability and coincided, with a small lead, with the worldwide crisis in financial markets that opened in the second half of October. The estimates for 1987, shown in greater detail in Figure 5, indicate that volatility rose from 2.0 on 14 October to 3.1 per cent on the 15th (the highest value in the last four years) and then fell to a more normal level on the 16th. The crisis in world financial markets broke out on "Black Monday", the 19th.

Figure 5

DIFFERENTIAL BETWEEN LONG-TERM ASYMPTOTIC AND INSTANTANEOUS INTEREST RATES
AND IMPLIED VOLATILITY OF THE INSTANTANEOUS RATE
(Raw data: Italian Treasury bond prices)

N.B. The instantaneous interest rate, the long-term asymptotic rate and the implied volatility were obtained by estimating the parameters of the Cox, Ingersoll and Ross model.

Finally, by assuming a zero risk premium ($\lambda = 0$), the parameters κ and μ of process can also be obtained from (4) $\phi_1, \phi_2 \in \phi_3$. The average speed of adjustment κ is equal to 0.24, with a standard deviation of 0.003. The average value μ of the instantaneous rate expected in the long term is equal to 11.13 per cent, with a standard deviation of 1.802 (Table 6). Figure 6 shows the changes in the two parameters over the period.

Figure 6

LONG-RUN EXPECTED VALUE OF THE INSTANTANEOUS INTEREST RATE
AND SPEED OF ADJUSTMENT
(Raw data: Italian Treasury bond prices)

N.B. The long-run expected value of the instantaneous interest rate and the speed of adjustment were obtained under the hypothesis of a null risk premium.

3.3 CIR model versus cubic splines

An additional test of the ability of the CIR model to provide an adequate description of market conditions was made by comparing the foregoing results with those obtainable using an inductive method. To this end, the yield curve was estimated by way of regression techniques based on cubic splines. This method, which involves interpolation with a series of third-degree polynomials, leaves a considerable degree of freedom with respect to the functional forms that the yield curve can have and therefore provides a good test of the validity of the constraints that the CIR model imposes on the form of the curve.¹⁸

The Figure 7 and Figure 8 show the yield curves estimated with the cubic splines method and the CIR model on the basis of the prices observed on 10 March 1989 (with settlement on 15 March) for the 49 listed Treasury bonds. For the sake of comparison the two figures also show the yields on Treasury bills issued on 15 March.

The curve obtained with the cubic splines method has a more complex form. Nonetheless, it can be seen from Figure 9, Figure 10, Table 8 and Table 9 that the differences between the market prices of the Treasury bonds listed on the secondary market and the theoretical values obtained [using equation (1)] from the estimated yield curves are very small in both cases.¹⁹

In addition, the curve obtained with the cubic splines method has a very steep positive slope for maturities of more than two years, which is reflected in highly erratic forward rates (Figure 12 and Table 11). More generally, the curves obtained with the cubic splines method are marked by a high degree of variability for longer maturities. The forward rates implicit in the curve estimated using the CIR model change much more smoothly (Figure 11 and Table 10). Since this curve is obtained on the basis of a general equilibrium model, it can also be extrapolated.

Throughout the period considered the differences between the actual and the theoretical prices calculated using the cubic splines method were slightly smaller than those obtained with the CIR model: 31 per cent of the absolute differences were less than 0.10 and 86 per cent less than 0.50 (Figure 13). By contrast, the configuration of the yield curve obtained with the cubic splines method is generally less stable, as can be seen from Table 7, which compares the variability of the daily estimates of the yields given by the two methods.

3.4 Assessment of the issue conditions of Italian government securities

Since the theoretical prices of bonds of any maturity can be determined on a daily basis with the CIR model, it can be used to test the consistency of the issue conditions of Treasury bills and bonds with

¹⁸ For a description of the *cubic splines* method see Appendix A.

¹⁹ The R^2 of the regression is not significantly different from 1 in either case and thus does not provide a criterion for choosing between the two methods.

the term structure of interest rates on the secondary market.²⁰

Figure 9

Figure 10

²⁰ For an application of the CIR model to the assessment of Treasury option certificates, see E. Barone and D. Cuoco (1989).

Figure 11

Figure 12

Figure 13

The average tender rate on Treasury bills in the auctions of the last five years have been compared with the estimates provided by the model for the same maturities. The value date of each estimate coincides with the subscription settlement date fixed for the corresponding auction. The results are shown in Figure 14.

It can be seen that, at least from 1986 on, interest rate conditions have been less favourable on average than those estimated by the model for the corresponding maturities. Furthermore, the size of the disparity appears to be directly related to the duration of securities: it was smaller, and in some periods favourable to subscription at auction, for 3-month bills; for 12-month bills it was much larger and always unfavourable to subscription at auction; and 6-month Treasury bills were in an intermediate position.

Figure 14

DIFFERENTIALS BETWEEN ACTUAL AND THEORETICAL YIELDS OF ITALIAN TREASURY BILLS

N.B. The theoretical yields were calculated on the basis of the net yield curve obtained by estimating the Cox, Ingersoll and Ross model.

A comparison has also been made between the issue and theoretical prices of Treasury bonds for the last five years and the results are shown in Figure 15.

Figure 15

DIFFERENTIALS BETWEEN ACTUAL AND THEORETICAL PRICES OF TREASURY BONDS AT ISSUE

N.B. The theoretical prices were calculated on the basis of the net yield curve obtained by estimating the Cox, Ingersoll and Ross model.

It is possible to distinguish three periods quite clearly: in the first, which lasted from 1984 to the middle of 1986, the issue conditions and those estimated by the model appear to be in line on average; in the second, from the middle of 1986 to the end of 1987, the issue conditions were less favourable; while in the third, from the beginning of 1988 to February 1989, the two prices converged again and there was even a tendency in the last part of 1988 for the Treasury to offer 4-year Treasury bonds at very favourable conditions.

4. CONCLUSIONS

The yield curves estimated on the basis of the CIR model have a more stable and less complex configuration than those obtained with inductive methods. Insofar as they derive from a theoretical model, they can be extrapolated beyond the range of maturities actually observed. Furthermore, the estimated values of the parameters of the model are of general interest, providing, for example, information on the expected future volatility of interest rates.

The daily estimation of the term structure of interest rates on the basis of the CIR model confirmed that it explains the observed variability extremely well. The mean value of the distribution of the differences between actual and theoretical prices (26,058 observations) is not significantly different from zero and 80 per cent are less than 0.50.

The changes in the estimated yield curve are consistent with the generally accepted interpretation of the developments in the period. In particular, at the end of June 1986, after the Italian stock market crisis in May and in conjunction with difficulties in selling government securities, the slope of the yield curve abruptly reversed, as a result of the sharp rise in long-term rates associated with the change in expectations. In addition, there was an increase in implicit volatility that slightly anticipated and accompanied the crisis in world financial markets that broke out in October 1987.

The model makes it possible to determine the theoretical prices of fixed rate securities of any maturity and therefore allows the efficiency of the secondary market to be assessed and the consistency between the primary and the secondary markets tested. A retrospective analysis of the interest conditions at Treasury bill auctions shows that, at least from 1986 on, these were less favourable than those estimated by the model. The disparity was particularly large in the case of 12-month bills. The issue conditions of Treasury bonds between the middle of 1986 and the end of 1987 were also unfavourable, while since the beginning of 1988 they appear to have been more in line with the estimates obtained with the model and tended to be extremely favourable in the case of 4-year bonds.

APPENDIX A

THE CUBIC SPLINES METHOD

The estimation of the yield curve by means of a regression based on the use of polynomial splines was first proposed by McCulloch (1971, 1975). It involves dividing the time axis into a series of intervals and approximating, within each interval, the function that links the price P to the maturity with a polynomial function. This technique leaves a considerable degree of freedom to the functional forms that the curve can have while still requiring the estimation of a linear regression. In the event that polynomials of the third degree (cubic splines), or of a higher degree, are used, the forward interest rates implicit in the curve are also linked to maturities by a continuous function.

To apply the cubic splines method to the estimate of the yield curve, it is assumed that the discount function can be approximated by a κ parameter function defined over $k - 2$ intervals between $k - 1$ “nodes” d_j ($j = 1, \dots, k - 1$).²¹ In other words:

$$P(t, T) = 1 + \sum_{j=1}^k \beta_j f_j(T - t; d_1, d_2, \dots, d_{k-1}). \quad (a1)$$

The f_j are polynomial functions in the argument $T - t$ of no more than the third degree. They define the so-called “basis” of the splines. For $P(T, T) = 1$ to be true, we must have $f_j(0) = 0$.

Substituting (a1) in (1), we have:

$$B(t, t_n) = \sum_{i=1}^n c_i P(t, t_i) = \sum_{i=1}^n c_i \left[1 + \sum_{j=1}^k \beta_j f_j(t_i - t) \right] = \alpha + \sum_{j=1}^k \beta_j x_j \quad (a2)$$

where:

$$\alpha = \sum_{i=1}^n c_i \quad \text{e} \quad x_j = \sum_{i=1}^n c_i f_j(t_i - t).$$

The β_j coefficients of (a2) can therefore be estimated by means of the following linear regression:

$$\mathbf{b} - \alpha = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \quad (a3)$$

G. S. Shea (1984) has shown that the regression (a3) based on cubic splines is equivalent to the estimation, using bounded least squares, of a series of $(k - 2)$ chained polynomials.²² The polynomials are constrained to make their ordinates and first and second derivatives equal in correspondence with each node. The resulting approximation of the discount function is thus continuous and twice differentiable. This ensures, in particular, that the implicit forward rates are a continuous function of the maturity.

One basic problem regarding the application of splines concerns the choice of the number and position of the “nodes” and the identification of the functional form of the f_j elements in the “basis”.

As for the choice of the number of “nodes”, $k - 1$, it should be noted that if k is too small, the approximation of the discount function will prove to be inadequate if the function has a complex configuration. On the other hand, if k based on cubic splines is equivalent to the estimation, using bounded least squares, of a series (in the benchmark case equal to the number of observations), the

²¹ The method of approximation based on splines has its theoretical basis in Weierstrass’s Theorem, according to which a function that is continuously differentiable can be approximated in any interval, with an arbitrarily small error, by a polynomial defined in the same interval. In other words, assuming the existence of a function of degree n that links the price P of a pure discount bond to its maturity, it is possible to find a polynomial of a degree less than n that approximates the function with sufficient accuracy in a given interval.

²² However, splines offer a more parsimonious solution: while the bounded least squares require the estimation of $4k - 8$ parameters, the cubic splines method only requires the estimation of k parameters. See Cf. G. S. Shea (1984), p. 259.

resulting function will be free to adapt exactly to any outliers. As a general rule, it appears reasonable to choose k so that it is an increasing function of the number of observations, n . This function should also be defined so that the number of observations falling within each interval, $n/(k-2)$, does not decrease as n increases. In the applications described in this paper k was made equal to $k = \sqrt{n}$.

In order to ensure that the estimation of the β_j parameters is sufficiently accurate, it appears advisable to choose the positions of the nodes $(d_1, d_2, \dots, d_{k-1})$ so as to have the same number of securities maturing in each interval. In other words, if T_i is the series, in increasing order, of the maturities of the n securities in the sample, we put $d_1 = 0$ and:

$$d_j = T_h + \theta(T_{h+1} - T_h) \quad (j = 2, \dots, k-1)$$

with

$$h = \text{il pi\`u grande intero in } \frac{(j-1)n}{k-2}$$

and

$$\theta = \frac{(j-1)n}{k-2} - h.$$

As for the choice of the f_j , the specification proposed in the Appendix of McCulloch (1975) was adopted.

As in the estimation of the CIR model, it was assumed that the residuals of (a3) were mutually independent and normally distributed, with a zero mean and variance proportional to the derivatives of the prices of the securities with respect to their yields to maturity. In addition, the procedure for excluding outliers described in §2.2 was also adopted.

APPENDIX B

TABLE 1 Italian Treasury Bonds listed on the Milan Stock Exchange and the Wire Market

<i>Date</i>	<i>Number of Securities</i>	<i>Par Value (billions of lire)</i>
30 Dec 83	7	15,890
31 Dec 84	8	29,019
31 Dec 85	9	23,056
31 Dec 86	21	49,582
31 Dec 87	33	70,803
30 Dec 88	47	112,527

TABLE 2 Parameters of the CIR model (daily estimates; 30.12.1983 - 13.3.1989)

ϕ_1				
<i>Period</i>	<i>Mean</i> (1)	<i>Std. dev.</i> (2)	<i>Var. coef.</i> (2)/(1)	<i>Range</i>
84-89*	0.25901	0.00104	0.00403	0.25633 - 0.26733
1984	0.25938	0.00098	0.00378	0.25780 - 0.26292
1985	0.25834	0.00065	0.00252	0.25633 - 0.26228
1986	0.25860	0.00065	0.00251	0.25676 - 0.26075
1987	0.25942	0.00136	0.00522	0.25825 - 0.26733
1988	0.25925	0.00109	0.00393	0.25839 - 0.26535
1989*	0.25923	0.00011	0.00042	0.25895 - 0.25945
ϕ_2				
<i>Period</i>	<i>Mean</i> (1)	<i>Std. dev.</i> (2)	<i>Var. coef.</i> (2)/(1)	<i>Range</i>
84-89*	0.25118	0.00104	0.00414	0.24618 - 0.25438
1984	0.25049	0.00132	0.00527	0.24618 - 0.25255
1985	0.25188	0.00077	0.00304	0.24757 - 0.25438
1986	0.25161	0.00067	0.00267	0.24968 - 0.25375
1987	0.25091	0.00105	0.00419	0.24669 - 0.25189
1988	0.25106	0.00070	0.00278	0.24735 - 0.25173
1989*	0.25092	0.00011	0.00043	0.25076 - 0.25125
ϕ_3				
<i>Period</i>	<i>Mean</i> (1)	<i>Std. dev.</i> (2)	<i>Var. coef.</i> (2)/(1)	<i>Range</i>
84-89*	14.271	2.8197	0.19758	5.753 - 49.625
1984	12.391	2.4953	0.20138	6.218 - 16.458
1985	15.595	3.7402	0.23984	7.626 - 49.625
1986	15.105	1.8723	0.12395	8.978 - 22.485
1987	13.507	2.5194	0.18653	5.753 - 15.835
1988	14.398	1.9975	0.13874	6.104 - 16.238
1989*	16.224	0.2962	0.01826	15.356 - 15.585

* The data on 1989 refer to the period January 2 - March 13.

TABLE 3 Instantaneous (r) and long-term asymptotic (r_l) interest rates

r			
<i>Period</i>	<i>Mean</i>	<i>Std. dev.</i> ¹	<i>Range</i>
84-89*	11.519	3.156	8.071 - 16.129
1984	14.948	2.702	13.295 - 16.129
1985	13.293	3.655	11.849 - 14.250
1986	10.754	4.759	8.179 - 13.623
1987	9.574	2.022	8.071 - 10.740
1988	9.423	1.945	8.914 - 9.988
1989*	9.466	1.989	9.026 - 10.665
r_l			
<i>Period</i>	<i>Mean</i>	<i>Std. dev.</i> ¹	<i>Range</i>
84-89*	10.774	11.333	4.199 - 16.187
1984	10.524	8.827	7.540 - 14.536
1985	9.794	14.319	4.255 - 14.180
1986	10.524	16.922	4.199 - 16.187
1987	10.968	5.386	8.520 - 13.765
1988	11.52	7.784	8.504 - 14.096
1989*	13.489	8.534	11.815 - 14.195

* The data on 1989 refer to the period January 2 - March 13.

¹ Annualized standard deviation of daily changes.

TABLE 4 Correlation coefficients between the instantaneous interest rate and some money market rates

Period	Interbank rates							3 month T-bills
	Overnight			Call	7 days	1 months	3 months	
	£	\$	DM					
84-89*	0.40	0.70	0.16	0.93	0.90	0.95	0.93	0.98
1984	0.30	-0.25	-	0.02	0.62	0.66	-	0.79
1985	-0.08	-0.22	-	0.14	-0.10	-0.06	0.15	0.54
1986	0.60	0.46	-	0.94	0.93	0.93	0.94	0.94
1987	0.08	0.58	-	-0.26	0.00	0.18	0.53	0.61
1988	0.05	0.16	0.12	0.10	0.07	0.07	0.09	0.22
1989*	0.11	0.76	0.44	0.87	0.53	0.79	0.84	0.26

* The data on 1989 refer to the period January 2 - March 13.

TABLE 5 Variability of the instantaneous interest rate: $\sigma\sqrt{r}$ (daily estimates; 30 December 1983 - 13 March 1989)

<i>Period</i>	<i>Mean</i>	<i>Std. dev.</i>	<i>Range</i>
84-89*	2.1004	0.33214	1.1606 - 3.5880
1984	2.5592	0.31846	1.9602 - 3.5880
1985	2.0653	0.21638	1.1606 - 3.0926
1986	1.9269	0.19903	1.3213 - 2.3080
1987	2.0088	0.2844	1.7432 - 3.1487
1988	1.9606	0.17685	1.8144 - 2.8818
1989*	1.9865	0.03968	1.9092 - 2.0761

* The data on 1989 refer to the period January 2 - March 13.

TABLE 6 Long-run expected value of the instantaneous interest rate (μ) and speed of adjustment (κ)

μ			
<i>Period</i>	<i>Mean</i>	<i>Std. dev.</i>	<i>Range</i>
84-89*	11.128	1.8023	4.261 - 16.837
1984	10.917	1.4255	7.705 - 15.069
1985	10.059	1.85815	4.322 - 14.760
1986	10.834	2.26646	4.261 - 16.837
1987	11.536	1.20238	8.741 - 14.260
1988	11.909	1.15063	8.874 - 14.605
1989*	13.951	0.56265	12.189 - 14.704
κ			
<i>Period</i>	<i>Mean</i>	<i>Std. dev.</i>	<i>Range</i>
84-89*	0.2433	0.00308	0.2269 - 0.2524
1984	0.2416	0.00361	0.2296 - 0.2473
1985	0.2454	0.00218	0.2329 - 0.2524
1986	0.2446	0.00199	0.2386 - 0.2507
1987	0.2424	0.00341	0.2269 - 0.2455
1988	0.2429	0.00238	0.2296 - 0.2451
1989*	0.2426	0.00032	0.2421 - 0.2436

* The data on 1989 refer to the period January 2 - March 13.

TABLE 7 Net interest rates (daily estimates; 30 December 1983 - 13 March 1989)

<i>Cox, Ingersoll and Ross model</i>			
<i>Maturity</i>	<i>Mean</i>	<i>Std. dev.</i>	<i>Range</i>
1 month	12.233	2.5594	8.454 - 17.462
3 months	12.222	2.4978	8.548 - 17.383
6 months	12.206	2.4094	8.683 - 17.321
1 year	12.173	2.2421	8.945 - 17.251
3 years	12.047	1.7507	9.695 - 16.989
5 years	11.939	1.4755	8.791 - 16.766
 <i>Cubic splines</i>			
<i>Maturity</i>	<i>Mean</i>	<i>Std. dev.</i>	<i>Range</i>
1 month	11.709	2.8855	6.101 - 17.112
3 months	11.771	2.769	6.942 - 17.144
6 months	11.887	2.6154	7.962 - 17.188
1 year	12.124	2.3124	8.712 - 17.256
3 years	12.061	1.7022	9.668 - 17.128
5 years	11.077	1.7812	7.175 - 17.668

TABLE 8 Differentials between actual and theoretical prices: Cox, Ingersoll and Ross model (Friday 10 March 1989)

<i>A.B.I. code</i>	<i>Security</i>	<i>Tax rate</i>	<i>Actual price</i>	<i>Theor. price</i>	<i>Differential</i>	<i>Stochastic duration</i>
12499	BTP 1.4.89 12.00%	-	99.95	99.93	0.02	0.05
12601	BTP 1.5.89 10.50%	-	100.00	99.88	0.12	0.13
12231	BOT 15.6.89 (3 m)	12.50	97.27	97.40	-0.13	0.25
12232	BOT 15.9.89 (6 m)	12.50	94.85	94.82	0.03	0.50
12492	BTP 1.1.90 12.50%	-	101.10	100.99	0.11	0.77
12610	BTP 1.1.90 9.25%	6.25	98.32	98.10	0.21	0.78
12495	BTP 1.2.90 12.50%	-	101.30	101.11	0.19	0.85
12612	BTP 1.2.90 9.25%	6.25	97.95	97.93	0.02	0.86
12497	BTP 1.3.90 12.50%	-	101.50	101.28	0.22	0.93
12614	BTP 1.3.90 9.15%	6.25	96.60	97.74	-1.14	0.94
12628	BTP 1.3.90 10.50%	12.50	98.35	98.21	0.14	0.94
12233	BOT 15.3.90 (12 m)	12.50	90.11	89.83	0.28	1.00
12629	BTP 15.3.90 10.50%	12.50	98.25	98.16	0.09	0.98
12600	BTP 1.4.90 12.00%	-	100.70	100.83	-0.13	0.96
12616	BTP 1.4.90 9.15%	6.25	97.23	97.49	-0.26	0.98
12630	BTP 1.4.90 10.50%	12.50	97.80	98.06	-0.26	0.98
12632	BTP 15.4.90 10.50%	12.50	97.95	97.91	0.04	1.02
12602	BTP 1.5.90 10.50%	-	99.15	99.33	-0.18	1.05
12618	BTP 1.5.90 9.15%	6.25	97.20	97.28	-0.08	1.06
12633	BTP 1.5.90 10.50%	12.50	97.80	97.87	-0.07	1.06
12635	BTP 18.5.90 10.50%	12.50	97.75	97.81	-0.06	1.10
12603	BTP 1.6.90 10.00%	-	98.45	98.72	-0.27	1.14
12620	BTP 1.6.90 9.15%	6.25	97.00	97.09	-0.09	1.15
12636	BTP 16.6.90 10.50%	12.50	97.70	97.58	0.12	1.19
12604	BTP 1.7.90 9.50%	-	97.60	98.00	-0.40	1.22
12622	BTP 1.7.90 10.50%	6.25	97.80	98.39	-0.59	1.22
12637	BTP 1.7.90 11.00%	12.50	97.60	98.06	-0.46	1.22
12605	BTP 1.8.90 9.50%	-	97.60	97.88	-0.28	1.31
12624	BTP 1.8.90 10.50%	6.25	98.00	98.25	-0.25	1.30
12639	BTP 1.8.90 11.00%	12.50	97.90	97.94	-0.04	1.31
12606	BTP 1.9.90 9.25%	-	97.25	97.35	-0.10	1.40
12625	BTP 1.9.90 11.25%	12.50	98.90	98.02	0.88	1.39
12641	BTP 1.9.90 11.50%	12.50	98.10	98.32	-0.22	1.39
12607	BTP 1.10.90 9.25%	6.25	96.10	96.37	-0.27	1.41
12626	BTP 1.10.90 11.50%	12.50	98.30	98.31	-0.01	1.39
12608	BTP 1.11.90 9.25%	6.25	96.55	96.12	0.43	1.49
12609	BTP 1.12.90 9.25%	6.25	96.40	95.90	0.50	1.58
12498	BTP 1.3.91 12.50%	-	103.00	102.00	1.00	1.77
12611	BTP 1.1.92 9.25%	6.25	92.95	93.18	-0.23	2.42
12613	BTP 1.2.92 9.25%	6.25	93.15	92.94	0.21	2.51
12627	BTP 1.2.92 11.00%	12.50	95.80	95.18	0.62	2.48
12615	BTP 1.3.92 9.15%	6.25	93.55	92.57	0.98	2.59
12617	BTP 1.4.92 9.15%	6.25	92.80	92.36	0.44	2.51
12631	BTP 1.4.92 11.00%	12.50	93.45	94.86	-1.41	2.47
12619	BTP 1.5.92 9.15%	6.25	93.50	92.07	1.43	2.60
12634	BTP 1.5.92 11.00%	12.50	94.55	94.59	-0.04	2.56
12621	BTP 1.6.92 9.15%	6.25	93.85	91.91	1.94	2.68
12623	BTP 1.7.92 10.50%	6.25	98.30	95.13	3.17	2.71
12638	BTP 1.7.92 11.50%	12.50	96.00	95.59	0.41	2.70
12640	BTP 1.8.92 11.50%	12.50	95.20	95.32	-0.12	2.79
12642	BTP 1.9.92 12.50%	12.50	97.10	97.81	-0.71	2.84
12644	BTP 1.10.92 12.50%	12.50	96.33	97.73	-1.40	2.70

TABLE 9 Differentials between actual and theoretical prices: Cubic splines (Friday 10 March 1989)

<i>A.B.I. code</i>	<i>Security</i>	<i>Tax rate</i>	<i>Actual price</i>	<i>Theor. price</i>	<i>Differential</i>	<i>Stochastic duration</i>
12499	BTP 1.4.89 12.00%	-	99.95	99.93	-0.02	0.05
12601	BTP 1.5.89 10.50%	-	100.00	99.88	0.02	0.13
12231	BOT 15.6.89 (3 m)	12.50	97.27	97.40	-0.32	0.25
12232	BOT 15.9.89 (6 m)	12.50	94.85	94.82	-0.29	0.50
12492	BTP 1.1.90 12.50%	-	101.10	100.99	-0.09	0.77
12610	BTP 1.1.90 9.25%	6.25	98.32	98.10	0.02	0.78
12495	BTP 1.2.90 12.50%	-	101.30	101.11	0.08	0.86
12612	BTP 1.2.90 9.25%	6.25	97.95	97.93	-0.09	0.86
12497	BTP 1.3.90 12.50%	-	101.50	101.28	0.19	0.93
12614	BTP 1.3.90 9.15%	6.25	96.60	97.74	-1.16	0.94
12628	BTP 1.3.90 10.50%	12.50	98.35	98.21	0.12	0.94
12233	BOT 15.3.90 (12 m)	12.50	90.11	89.83	0.32	1.00
12629	BTP 15.3.90 10.50%	12.50	98.25	98.16	0.11	0.98
12600	BTP 1.4.90 12.00%	-	100.70	100.83	-0.06	0.97
12616	BTP 1.4.90 9.15%	6.25	97.23	97.49	-0.19	0.99
12630	BTP 1.4.90 10.50%	12.50	97.80	98.06	-0.18	0.98
12632	BTP 15.4.90 10.50%	12.50	97.95	97.91	0.15	1.02
12602	BTP 1.5.90 10.50%	-	99.15	99.33	-0.04	1.06
12618	BTP 1.5.90 9.15%	6.25	97.20	97.28	0.06	1.07
12633	BTP 1.5.90 10.50%	12.50	97.80	97.87	0.07	1.07
12635	BTP 18.5.90 10.50%	12.50	97.75	97.81	0.11	1.11
12603	BTP 1.6.90 10.00%	-	98.45	98.72	-0.08	1.14
12620	BTP 1.6.90 9.15%	6.25	97.00	97.09	0.10	1.15
12636	BTP 16.6.90 10.50%	12.50	97.70	97.58	0.32	1.19
12604	BTP 1.7.90 9.50%	-	97.60	98.00	-0.21	1.23
12622	BTP 1.7.90 10.50%	6.25	97.80	98.39	-0.40	1.23
12637	BTP 1.7.90 11.00%	12.50	97.60	98.06	-0.26	1.23
12605	BTP 1.8.90 9.50%	-	97.60	97.88	-0.11	1.31
12624	BTP 1.8.90 10.50%	6.25	98.00	98.25	-0.09	1.31
12639	BTP 1.8.90 11.00%	12.50	97.90	97.94	0.12	1.31
12606	BTP 1.9.90 9.25%	-	97.25	97.35	0.00	1.40
12625	BTP 1.9.90 11.25%	12.50	98.90	98.02	0.98	1.40
12641	BTP 1.9.90 11.50%	12.50	98.10	98.32	-0.12	1.40
12607	BTP 1.10.90 9.25%	6.25	96.10	96.37	-0.24	1.42
12626	BTP 1.10.90 11.50%	12.50	98.30	98.31	0.01	1.41
12608	BTP 1.11.90 9.25%	6.25	96.55	96.12	0.34	1.51
12609	BTP 1.12.90 9.25%	6.25	96.40	95.90	0.30	1.60
12498	BTP 1.3.91 12.50%	-	103.00	102.00	0.39	1.80
12611	BTP 1.1.92 9.25%	6.25	92.95	93.18	-1.18	2.49
12613	BTP 1.2.92 9.25%	6.25	93.15	92.94	-0.62	2.58
12627	BTP 1.2.92 11.00%	12.50	95.80	95.18	-0.22	2.56
12615	BTP 1.3.92 9.15%	6.25	93.55	92.57	0.28	2.66
12617	BTP 1.4.92 9.15%	6.25	92.80	92.36	-0.09	2.63
12631	BTP 1.4.92 11.00%	12.50	93.45	94.86	-1.95	2.58
12619	BTP 1.5.92 9.15%	6.25	93.50	92.07	1.13	2.72
12634	BTP 1.5.92 11.00%	12.50	94.55	94.59	-0.36	2.68
12621	BTP 1.6.92 9.15%	6.25	93.85	91.91	1.85	2.80
12623	BTP 1.7.92 10.50%	6.25	98.30	95.13	3.32	2.84
12638	BTP 1.7.92 11.50%	12.50	96.00	95.59	0.56	2.82
12640	BTP 1.8.92 11.50%	12.50	95.20	95.32	0.34	2.91
12642	BTP 1.9.92 12.50%	12.50	97.10	97.81	0.06	2.96
12644	BTP 1.10.92 12.50%	12.50	96.33	97.73	-0.28	2.88

TABLE 10 Theoretical prices and net interest rates: Cox, Ingersoll and Ross model (Friday 10 March 1989)

<i>Maturity</i>	<i>Theoretical price</i>	<i>Spot rate</i>	<i>Forward rate</i>		
			<i>3 months</i>	<i>6 months</i>	<i>12 months</i>
1 month	99.15	10.95	11.09	11.20	11.38
3 months	97.45	11.02	11.23	11.33	11.50
6 months	94.79	11.13	11.43	11.52	11.68
1 year	89.83	11.32	11.76	11.85	11.99
2 years	80.22	11.65	12.32	12.38	12.49
3 years	71.31	11.93	12.74	12.79	12.87

Raw data: Italian Treasury bonds listed on the Milan Stock Exchange and the Wire Market.

TABLE 11 Theoretical prices and net interest rates: Cubic splines (Friday 10 March 1989)

<i>Maturity</i>	<i>Theoretical price</i>	<i>Spot rate</i>	<i>Forward rate</i>		
			<i>3 months</i>	<i>6 months</i>	<i>12 months</i>
1 month	99.21	10.19	10.23	10.59	11.59
3 months	97.64	10.18	10.58	11.21	11.97
6 months	95.11	10.40	11.85	12.38	12.16
1 year	89.80	11.36	12.65	11.97	11.09
2 years	80.84	11.22	10.44	11.05	12.52
3 years	71.84	11.66	16.80	17.93	-

Raw data: Italian Treasury bonds listed on the Milan Stock Exchange and the Wire Market.

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